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EXISTENCE THEOREM FOR A FRACTAL STURM–LIOUVILLE PROBLEM

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Abstract. In this article, using a new calculus defined on fractal subsets of the set of real numbers, a Sturm–Lioville type problem is discussed, namely the fractal Sturm–Liouville problem. The existence and uniqueness theorem has been proved for such equations. In this context, the historical development of the subject is discussed in the introduction. In Section 2, the basic concepts of F^{α}-calculus defined on fractal subsets of real numbers are given, i. e., F^{α}-continuity, F^{α}-derivative and fractal integral definitions are given and some theorems to be used in the article are given. In Section 3, the existence and uniqueness of the solutions for the fractal Sturm–Liouville problem are obtained by using the successive approximations method. Thus, the well-known existence and uniqueness problem for Sturm–Liouville equations in ordinary calculus is handled on the fractal calculus axis, and the existing results are generalized.

Keywords: fractal Sturm–Liouville problems, existence problems.

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1. Introduction

It is well-known that fractal calculus is a generalization of ordinary calculus. It is applied for functions which are not differentiable on totally disconnected fractal sets. In 2009, Parvate and Gangal [1] defined the concept of F^{α}-calculus on fractal subsets of real numbers. Later, the relation of the F^{α}-integral and F^{α}-derivative with classical Riemann integral and ordinary derivative is investigated. Although the fractional derivative is not local, the F^{α}-derivative is local and has many of the properties of the classical derivative. Due to these advantages, many researchers are working on this subject (see [2–7]). In [4], Golmankhaneh and Tunç studied a fractal stochastic differential equation. In [5], the authors studied the Laplace and Sumudu transforms in F^{α}-calculus. The existence and uniqueness theorems for the linear and non-linear fractal differential equations are proved in [6]. Recently, Çetinkaya and Golmankhaneh studied a regular fractal Sturm–Liouville problems [2] defined by

$$-\left(D_F^{\alpha}\right)^2 y(x)+q(x)y(x)=\lambda y(x), \quad x\in[0,\pi]\,, \tag{1}$$

where λ is a complex parameter, $q(.)$ is a real-valued function and $q\in L_2^{\alpha}[0,\pi]$. They proved some spectral properties of Eq. (1).

On the other hand, there are some earlier articles of Kolwankar and Gangal concerning local fractional derivatives exist [8–11]. In [8], Kolwankar and Gangal introduced the notion of local fractional derivative. Later they extended this definition to directional-local fractional derivatives for functions of many variables [11]. In [9], they reviewed a more direct method to characterize the local behaviour of functions. The authors studied a local fractional analog of the Fokker-Planck equation in [10].

As known, Sturm–Liouville problems defined as

$$-\frac{d}{dx}\left(p\frac{dy}{dx}\right)+qy=\lambda y,\quad -\infty<a\leqslant x\leqslant b<\infty,$$

are one of the most studied differential equations in the literature. Especially when solving partial differential equations with the Fourier method, its importance has increased even more. Such problems are investigated in various situations and boundary conditions (see [12–16]). In [12, 13, 15], the authors studied some Sturm–Liouville problems with impulsive conditions. The inverse problem for the Sturm–Liouville equation in the discrete state is worked out in [17]. A Sturm–Liouville problem with some nonlocal boundary conditions is studied in [16]. The q-Sturm–Liouville problems obtained by taking the q-derivative instead of the classical derivative in the Sturm–Liouville equation were studied in [18, 19].

In the present article, an existence theorem for Eq. (1) is obtained on the interval $[0,1]$. Thus, the well-known existence and uniqueness problem for Sturm–Liouville equations in ordinary calculus is handled on the fractal calculus axis, and the existing results are generalized.

2. Preliminaries

In this section, our goal is to present some basic concepts concerning the theory of fractal calculus (see [1, 2, 7]). Throughout the paper, we let F is a fractal subset of real numbers.

Definition 1 [1]. Given a partition

$$P_I=\{a=x_0<x_1<x_2<\ldots<x_n=b\}$$

of $I=[a,b]\subset\mathbb{R}$, we define $\sigma^{\alpha}[F,P]$ by

$$\sigma^{\alpha}[F,P]=\sum_{i=0}^{n-1}\frac{(x_{i+1}-x_i)^{\alpha}}{\Gamma(\alpha+1)}\theta\big(F,[x_i,x_{i+1}]\big),$$

where $0<\alpha\leqslant 1$ and

$$\theta(F,I)=\begin{cases}1, & \text{if } F\cap I\neq\varnothing,\\ 0, & \text{otherwise.}\end{cases}$$

If $a=b$, $\sigma^{\alpha}[F,P]=0$.

Definition 2 [1]. Let $\delta>0$ and $a\leqslant b$. Then the coarse-grained mass $\gamma_{\delta}^{\alpha}(F,a,b)$ is defined by

$$\gamma_{\delta}^{\alpha}(F,a,b)=\inf_{\{P_I:\,|P|\leqslant\delta\}}\sigma^{\alpha}[F,P],$$

where

$$|P|=\max_{0\leqslant i\leqslant n-1}(x_{i+1}-x_i)$$

for a partition P.

Definition 3 [1]. Let
$$\gamma^{\alpha}(F,a,b)=\lim_{\delta\to 0}\gamma^{\alpha}_{\delta}(F,a,b).$$

Definition 4 [1]. The γ-dimension of $F\cap I$ is given as
$$\dim_{\gamma}(F\cap I)=\inf\left\{\alpha:\gamma^{\alpha}(F,a,b)=0\right\}=\sup\left\{\alpha:\gamma^{\alpha}(F,a,b)=\infty\right\}.$$

Definition 5 [1].Let $f:\mathbb{R}\to\mathbb{R}$ be a function and $x\in F$, where $F\subset\mathbb{R}$. Then the F-limit of f is A, i. e.,
$$A=F-\lim_{y\to x}f(y)$$
if and only for any chosen positive number ε, however small, there exists positive number δ such that, whenever $|y-x|<\delta$, then
$$|f(y)-A|<\varepsilon.$$

Definition 6 [1]. A function f is the F-continuous at $x\in F$ if
$$F-\lim_{y\to x}f(y)=f(x)$$
holds.

Definition 7 [1]. Let $x\in(\eta,\beta)\subset\mathbb{R}$. If a function f is not constant over (η,β), then x is called the point of change of f. Let
$$\mathrm{Sch}(f)=\{x:x\text{ is the point of change of }f\}.$$

Then $\mathrm{Sch}\left(S^{\alpha}_{F}\right)$ is called α-perfect set, if $\mathrm{Sch}\left(S^{\alpha}_{F}\right)$ is a closed and every point of it is a limit point.

Definition 8 [1]. Let
$$S^{\alpha}_{F}(x)=\begin{cases}\gamma^{\alpha}(F,a_0,x), & \text{if } x\geqslant a_0,\\ -\gamma^{\alpha}(F,x,a_0), & \text{otherwise,}\end{cases}$$
where a_0 is arbitrary and fixed real number and let F be an α-perfect set. Then the F^{α}-derivative of f is given by
$$D^{\alpha}_{F}f(x)=\begin{cases}F-\lim\frac{f(y)-f(x)}{S^{\alpha}_{F}(y)-S^{\alpha}_{F}(x)}, & \text{if } x\in F,\\ 0, & \text{otherwise,}\end{cases}$$
if the limit exists.

Theorem 1 [1]**.** *If the functions $f,g:\mathbb{R}\to\mathbb{R}$ are F^{α}-differentiable, then we have*
$$\begin{aligned}&i)\ D^{\alpha}_{F}(fg)(x)=g(x)D^{\alpha}_{F}f(x)+f(x)D^{\alpha}_{F}g(x),\\ &ii)\ D^{\alpha}_{F}(af+bg)(x)=aD^{\alpha}_{F}f(x)+bD^{\alpha}_{F}g(x),\end{aligned}$$
where $a,b\in\mathbb{R}$.

Theorem 2 [1]**.** *The definite fractal integral of the function f is given by*
$$g(x)=\int_{a}^{x}f(y)\,d^{\alpha}_{F}y,$$

where $x \in I$, f is a bounded function on $F \cap I$, and

$$D_F^\alpha \int_a^x f(y)\, d_F^\alpha y = f(x)\, \chi_F(x).$$

Theorem 3 [1]**.** *Suppose $h : \mathbb{R} \to \mathbb{R}$ is a F^α-differentiable function and $\mathrm{Sch}(f)$ is contained in an α-perfect set F. Then we have*

$$\int_a^b g(x)\, d_F^\alpha x = h(b) - h(a),$$

where $g : \mathbb{R} \to \mathbb{R}$ is a F-continuous function and

$$g(x)\chi_F(x) = D_F^\alpha h(x).$$

Theorem 4 [1]**.** *Let $f, g \in C[a,b]$, where $\mathrm{Sch}(f) \subset F$. Moreover $D_F^\alpha f$ exists and is F-continuous on I. Then we have*

$$\int_a^b f(x)\, g(x)\, d_F^\alpha x + \int_a^b D_F^\alpha f(x) \int_a^x g(y)\, d_F^\alpha\, y\, d_F^\alpha x = \left[f(x) \int_a^x g(y)\, d_F^\alpha y \right]_a^b.$$

Let $\mathrm{Sch}(f)$ be an α-perfect set and let

$$L_2^\alpha[0,1] = \left\{ f : \int_0^1 |f(x)|^2\, d_F^\alpha x < \infty \right\}.$$

Then $L_2^\alpha[0,1]$ is a Hilbert space endowed with the inner product

$$(f,g) = \int_0^1 fg\, d_F^\alpha x.$$

The α-Wronskian [2] of f, g is defined to be

$$W^\alpha(x) = f(x)\, D_F^\alpha g(x) - g(x)\, D_F^\alpha f(x), \quad x \in [0,1].$$

3. Main Results

Let us consider the following fractal Sturm–Liouville equation

$$-\left(D_F^\alpha\right)^2 y(x) + q(x)y(x) = \lambda y(x), \quad x \in [0,1], \tag{2}$$

where λ is a complex parameter, $q(\cdot)$ is a real valued-function and $q \in L_2^\alpha[0,1]$.

Theorem 5. *Eq.* (2) *has a unique solution Ψ in $L_2^\alpha[0,1]$ which satisfies*

$$\Psi(0,\lambda) = c_1,\ D_F^\alpha \Psi(0,\lambda) = c_2, \quad c_1, c_2, \lambda \in \mathbb{C}. \tag{3}$$

◁ Let

$$\Phi_1(x,\lambda) = \cos\left(\sqrt{\lambda} S_F^\alpha(x)\right)$$

and

$$\Phi_2(x,\lambda) = \begin{cases} \frac{\sin\left(\sqrt{\lambda} S_F^\alpha(x)\right)}{\sqrt{\lambda}}, & \lambda \neq 0, \\ S_F^\alpha(x), & \lambda = 0, \end{cases} \tag{4}$$

be a fundamental system of

$$(D_F^\alpha)^2 + \lambda y = 0, \tag{5}$$

with $W^\alpha(\Phi_1(t,\lambda), \Phi_2(t,\lambda)) \neq 0$.

Let us construct a sequence $[\Xi_m(x,\lambda)]_{m=1}^\infty$ of successive approximations method by

$$\Xi_1(x,\lambda) = c_1\Phi_1(x,\lambda) + c_2\Phi_2(x,\lambda),$$

$$\begin{aligned} \Xi_{m+1}(x,\lambda) &= c_1\Phi_1(x,\lambda) + c_2\Phi_2(x,\lambda) \\ &+ \int_0^x \Big[\Phi_2(x,\lambda)\Phi_1(t,\lambda) - \Phi_1(x,\lambda)\Phi_2(t,\lambda)\Big] q(t)\Xi_m(t,\lambda)\chi_F(t)\, d_F^\alpha t \\ &= c_1\cos(\sqrt{\lambda} S_F^\alpha(x)) + c_2\frac{\sin\left(\sqrt{\lambda} S_F^\alpha(x)\right)}{\sqrt{\lambda}} \\ &+ \int_0^x \begin{bmatrix} \frac{\sin\left(\sqrt{\lambda} S_F^\alpha(x)\right)}{\sqrt{\lambda}}\cos(\sqrt{\lambda} S_F^\alpha(t)) \\ -\cos(\sqrt{\lambda} S_F^\alpha(x))\frac{\sin\left(\sqrt{\lambda} S_F^\alpha(t)\right)}{\sqrt{\lambda}} \end{bmatrix} q(t)\Xi_m(t,\lambda)\chi_F(t)\, d_F^\alpha t, \end{aligned} \tag{6}$$

where $x \in [0,1]$ and $\lambda \in \mathbb{C}$.

Let $\lambda \in \mathbb{C}$ be fixed. Using Weierstrass M-test, we next claim that the uniform limit of $\{\Xi_m(\cdot,\lambda)\}_{m=1}^\infty$ as $m \to \infty$ exists and defines the solution of (2)–(3). Let $|q(x)| \leqslant A$, $|\Phi_i(x,\lambda)| \leqslant \sqrt{\frac{\eta(\lambda)}{2}}$ $(i=1,2)$, for $0 \leqslant x \leqslant 1$, and let $\Xi_1(x,\lambda) \leqslant \widetilde{\eta(\lambda)}$. Then

$$\begin{aligned} |\Xi_2(x,\lambda) - \Xi_1(x,\lambda)| &\leqslant \left| \int_0^x \frac{\sin\left(\sqrt{\lambda} S_F^\alpha(x)\right)}{\sqrt{\lambda}} \cos(\sqrt{\lambda} S_F^\alpha(t)) q(t)\Xi_1(t,\lambda)\chi_F(t)\, d_F^\alpha t \right| \\ &+ \left| \int_0^x \cos(\sqrt{\lambda} S_F^\alpha(x)) \frac{\sin\left(\sqrt{\lambda} S_F^\alpha(t)\right)}{\sqrt{\lambda}} q(t)\Xi_1(t,\lambda)\chi_F(t)\, d_F^\alpha t \right| \\ &\leqslant (\widetilde{\eta(\lambda)} A \frac{\eta(\lambda)}{2} + \widetilde{\eta(\lambda)} A \frac{\eta(\lambda)}{2} \left| \int_0^x \chi_F(t)\, d_F^\alpha(x) \right| = \widetilde{\eta(\lambda)} A \eta(\lambda) \left| \int_0^x \chi_F(t)\, d_F^\alpha(x) \right|. \end{aligned} \tag{7}$$

Hence

$$|\Xi_2(x,\lambda) - \Xi_1(x,\lambda)| \leqslant \eta(\lambda) A \widetilde{\eta(\lambda)} S_F^\alpha(x),$$

and so generally

$$|\Xi_3(x,\lambda)-\Xi_2(x,\lambda)|\leqslant\frac{\left(\widetilde{\eta(\lambda)}\right)^2 A^2\eta^2(\lambda)\left(S_F^\alpha(x)\right)^2}{2!},$$

$$|\Xi_4(x,\lambda)-\Xi_3(x,\lambda)|\leqslant\frac{\left(\widetilde{\eta(\lambda)}\right)^3 A^3\eta^3(\lambda)\left(S_F^\alpha(x)\right)^3}{3!},$$

$$\ldots$$

$$|\Xi_{m+1}(x,\lambda)-\Xi_m(x,\lambda)|\leqslant\frac{\left(\widetilde{\eta(\lambda)}\right)^m A^m\eta^m(\lambda)\left(S_F^\alpha(x)\right)^m}{m!}\quad (m=1,2,\ldots).$$

It follows from Weierstrass M-test that the series

$$\Xi_1(x,\lambda)+\sum_{m=1}^{\infty}[\Xi_{n+1}(x,\lambda)-\Xi_m(x,\lambda)] \tag{8}$$

converge uniformly with respect to λ due to the series

$$\sum_{m=1}^{\infty}\frac{\left(\widetilde{\eta(\lambda)}\right)^m A^m\eta^m(\lambda)\left(S_F^\alpha(x)\right)^m}{m!}$$

are convergent. Then the n-th partial sum

$$\Xi_n(x,\lambda)=\Xi_1(x,\lambda)+\sum_{m=1}^{n-1}[\Xi_{m+1}(x,\lambda)-\Xi_m(x,\lambda)]$$

of this series approaches a function $\Psi(x,\lambda)$ uniformly on $[0,1]$ as $n\to\infty$, where $\Psi(x,\lambda)$ is the sum series and

$$\lim_{n\to\infty}\Xi_n(x,\lambda)=\Psi(x,\lambda).$$

By the uniform convergence, letting $m\to\infty$ in (6), we see that

$$\begin{gathered}\Psi(x,\lambda)=c_1\Phi_1(x,\lambda)+c_2\Phi_2(x,\lambda)\\ +\int_0^x\Big[\Phi_2(x,\lambda)\Phi_1(t,\lambda)-\Phi_1(x,\lambda)\Phi_2(t,\lambda)\Big]q(t)\Psi(t,\lambda)\chi_F(t)\,d_F^\alpha t.\end{gathered} \tag{9}$$

It is clear that Ψ satisfies (2) and (3).

We next claim that (2)–(3) has unique solution. Conversely, $\varphi_1(x,\lambda)$ and $\varphi_2(x,\lambda)$ are two solutions of (2)–(3). Define

$$\Lambda(x,\lambda)=\varphi_1(x,\lambda)-\varphi_2(x,\lambda),$$

where $x\in[0,1]$. Then Λ is a solution of (2). It is evident that

$$\Lambda(0,\lambda)=D_F^\alpha\Lambda(0,\lambda)=0.$$

By integrating twice to Eq. (2), one gets

$$\Lambda(x,\lambda)=\int_0^x\left[\int_0^t (q(s)-\lambda)\Lambda(s,\lambda)\chi_F(s)\,d_F^{\alpha}s\right]d_F^{\alpha}t.$$

Let $N_\lambda=\sup_{0\leqslant x\leqslant 1}|\Lambda(x,\lambda)|$, and $M_\lambda=\sup_{0\leqslant x\leqslant 1}|q(x)-\lambda|$. Then we obtain

$$|\Lambda(x,\lambda)|=\left|\int_0^x\left[\int_0^t (q(s)-\lambda)\Lambda(s,\lambda)\chi_F(s)\,d_F^{\alpha}s\right]d_F^{\alpha}t\right|$$
$$\leqslant N_\lambda M_\lambda\left|\int_0^x\left[\int_0^t \chi_F(s)\,d_F^{\alpha}s\right]d_F^{\alpha}t\right|=N_\lambda M_\lambda\frac{\left(S_F^{\alpha}(x)\right)^2}{2!}.$$

We shall prove it by mathematical induction. We assume that

$$|\Lambda(x,\lambda)|\leqslant N_\lambda^k M_\lambda^k\frac{\left(S_F^{\alpha}(x)\right)^{2k}}{(2k)!},\quad x\in[0,1]. \tag{10}$$

holds for a given $k\in\mathbb{N}$, and we prove it is true for $k+1$. Hence

$$|\Lambda(x,\lambda)|=\left|\int_0^x\left(\int_0^t (q(s)-\lambda)\Lambda(s,\lambda)\chi_F(s)\,d_F^{\alpha}s\right)d_F^{\alpha}t\right|$$
$$\leqslant N_\lambda^{k+1}M_\lambda^{k+1}\left|\int_0^x\left(\int_0^t \frac{\left(S_F^{\alpha}(s)\right)^{2k}}{(2k)!}\chi_F(s)\,d_F^{\alpha}s\right)d_F^{\alpha}t\right|\leqslant N_\lambda^{k+1}M_\lambda^{k+1}\frac{\left(S_F^{\alpha}(x)\right)^{2(k+1)}}{(2k+2)!}.$$

Consequently, we see that $\Lambda(x,\lambda)=0$ for all $x\in[0,1]$ due to

$$\lim_{k\to\infty}N_\lambda^{k+1}M_\lambda^{k+1}\frac{\left(S_F^{\alpha}(x)\right)^{2(k+1)}}{(2k+2)!}=0.$$

This concludes the proof. ▷

Conclusion. We considered a fractal Sturm–Liouville equation. We applied the classical method of the successive approximations in the theory of differential equations to fractal differential equations. Thus, the existence and uniqueness theorem for the fractal Sturm–Liouville problem is proved.

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ТЕОРЕМА СУЩЕСТВОВАНИЯ ФРАКТАЛЬНОЙ ЗАДАЧИ ШТУРМА — ЛИУВИЛЛЯ

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Аннотация. В этой статье, используя новое исчисление, определенное на фрактальных подмножествах множества действительных чисел, обсуждается вариант проблемы Штурма — Лиувилля, а именно фрактальная проблема Штурма — Лиувилля. Для таких уравнений доказана теорема существования и единственности. В этом контексте во введении обсуждается историческое развитие темы. Во втором параграфе представлены основные понятия F^{α}-исчисления, определенные на фрактальных подмножествах множества действительных чисел. Даны определения F^{α}-непрерывности, F^{α}-производной и фрактального интеграла, а также некоторые теоремы, которые используются в статье. В третьем параграфе получены существование и единственность решения фрактальной задачи Штурма — Лиувилля с помощью метода последовательных приближений. Таким образом, на оси фрактального исчисления решается классическая проблема существования и единственности для уравнения Штурма — Лиувилля, при этом обобщаются существующие результаты.

Ключевые слова: фрактальные проблемы Штурма — Лиувилля, проблемы существования.

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