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DYNAMICS OF QUADRATIC VOLTERRA-TYPE STOCHASTIC OPERATORS CORRESPONDING TO STRANGE TOURNAMENTS

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Abstract. By studying the dynamics of these operators on the simplex, focusing on the presence of an interior fixed point, we investigate the conditions under which the operators exhibit nonergodic behavior. Through rigorous analysis and numerical simulations, we demonstrate that certain parameter regimes lead to nonergodicity, characterized by the convergence of initial distributions to a limited subset of the simplex. Our findings shed light on the intricate dynamics of quadratic stochastic operators with interior fixed points and provide insights into the emergence of nonergodic behavior in complex dynamical systems. Also, the nonergodicity of quadratic stochastic operators of Volterra type with an interior fixed point defined in a simplex introduces additional complexity to the already intricate dynamics of such systems. In this context, the presence of an interior fixed point within the simplex further complicates the exploration of the state space and convergence properties of the operator. In this paper, we give sufficiency and necessary conditions for the existence of strange tournaments. Also, we prove the nonergodicity of quadratic stochastic operators of Volterra type with an interior fixed point, defined in a simplex.

Keywords: quadratic stochastic operators of Volterra type, simplex, strange tournaments, Lyapunov functions.

AMS Subject Classification: 37E99.

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1. Preliminaries

It is known that there are many systems which are described by nonlinear operators. One of the simplest nonlinear case is quadratic one. Quadratic dynamical systems have been proved to be a rich source of analysis for the investigation of dynamical properties and modeling in different domains, such as population dynamics (see [1, 2]) physics, mathematics (see [3]). On the other hand, the theory of Markov processes is a rapidly developing field with numerous applications to many branches of mathematics and physics [4, 5]. However, there are physical and biological systems that cannot be described by Markov processes. One of such system is given by quadratic stochastic operators (QSO), which are related to population genetics. The problem of studying the behavior of trajectories of quadratic stochastic operators was stated in [6]. The limit behavior and ergodic properties of trajectories of quadratic stochastic operators and their applications to population genetics were studied.

However, such kind of operators and processes do not cover the case of quantum systems. Therefore, in [7, 9] quantum quadratic operators acting on a von Neumann algebra were defined and studied. Certain ergodic properties of such operators were studied in [8]. In these papers, dynamics of quadratic operators were basically defined due to some recurrent rule which marks a possibility to study asymptotic behaviors of such operators.

This paper is devoted to the study of ergodic properties of quadratic stochastic operators of Volterra type defined in the standard simplex. The study of the dynamics of such operators began with the example of Ulam [6]. The non-ergodicity of this operator was proved by Zakharevich [10]. For a general form of quadratic stochastic operators of Volterra type, a proof of non-ergodicity in a two-dimensional simplex can be found, for example, in [11]. In the present paper, we prove the non-ergodicity of quadratic stochastic operators of Volterra type with an internal fixed point. In addition, the notion of a strange tournament is generalized to the case of r-strange tournaments. The application of the non-ergodicity of such operators in genetics is shown.

Let

$$S^{m-1}=\left\{x\in R^m : x=(x_1,\dots x_m) : x_i\geqslant 0, \sum_{i=1}^m x_i=1\right\}$$

be a standard $(m-1)$-dimensional simplex. Put

$$P_{ij,k}\geqslant 0,\quad P_{ij,k}=P_{ji,k},\quad \sum_{i,j=1}^m P_{ij,k}=1,\quad i,j,k=1,\dots,m. \tag{1.1}$$

For any $x\in S^{m-1}$ and for all $k=1,\dots,m$ we consider a mapping $V: S^{m-1}\Rightarrow S^{m-1}$ which is defined by

$$(Vx)_k=\sum_{i,j=1}^m P_{ij,k}x_ix_j. \tag{1.2}$$

Such operator is called a quadratic stochastic operator (Q.S.O.).

For $x^0=(x_1^0,x_2^0,\dots,x_m^0)\in S^{m-1}$ trajectory (orbit) is a sequence of iterations x^0, Vx^0, V^2x^0, …, V^nx^0, $n=0,1,2,\dots$

(Q.S.O.) V is called *regular* if for any $x\in \operatorname{int} S^{m-1}$ there exists a unique fixed point $x^*\in S^{m-1}$ (i. e. $Vx^*=x^*$) such that $\lim_{n\to\infty}V^nx=x^*$.

(Q.S.O.) V is called *ergodic* if for any $x\in S^{m-1}$ there exists a limit

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}V^kx.$$

A continuous functional $\varphi: S^{m-1}\to\mathbb{R}$ is called a Lyapunov function if for any initial point $x^{(0)}\in S^{m-1}$ there exists

$$\lim_{n\to\infty}\varphi\big(x^{(n)}\big).$$

(Q.S.O.) V defined on S^{m-1} is called a Q.S.O. Volterra type (Q.S.O.V.T.), if

$$P_{ij,k} = 0, \quad k \notin \{i, j\}.$$

(Q.S.O.V.T.) V can be reduced to the form

$$V : x'_k = x_k\left(1 + \sum_{i=1}^{m} a_{ki} x_i\right), \quad k = 1, \ldots, m, \tag{1.3}$$

where $|a_{ki}| \leqslant 1$, $a_{ki} = -a_{ik}$.

2. Tournaments. Strange Tournaments

Let $a_{ki} \neq 0$ for $k \neq i$. Consider a complete graph with m vertices labeled $1, 2, \ldots, m$. On the edges of the graph, we define directions as the following way: the edge connecting vertices k and i is directed from k-th vertex to i-th if $a_{ki} < 0$, and has the opposite direction, if $a_{ki} > 0$.

The resulting complete directed (oriented) graph is called a *tournament* and is denoted by T_m. Two tournaments T'_m and T''_m are called *isomorphic* if the graphs are isomorphic graphs.

A tournament is called *strong* if it is possible to get from any vertex to any other, taking into account the direction of the edges.

The transitivity of a tournament means that any subtournament of the given tournament is not strong.

Stock of a transitive tournament is a vertex from which it is impossible to get to any other vertex, taking into account the direction on the edges. Triangle is a strong tournament with three vertices.

Let us recall the following well-known results:

Denote by $S(k)$ the number of arcs entering to the k-th vertex, while $S(k)$ is called the number of points of the k-th vertex. The set of numbers $S(1)$, $S(2)$, …, $S(m)$ is called the order of the tournament.

A tournament T_m is called *strange* if there exists a (strange) vertex i_0 such that

1. $S(i_0) \neq S(k)$ for all $k \neq i_0$.

2. If $S(k) > S(i_0)$ for some k, then the arrow connecting vertices i_0 and k is directed from k to i_0.

3. If $S(k) < S(i_0)$, then the arrow has a direction from i_0 to k.

It is easy to check that there are no any *strange* tournaments in $T_m, (m = 2, 3, \ldots, 6)$.

A tournament T_m is called *r-strange* if there exist vertices $i_1, i_2, \ldots, i_r$ such that

1. $S(i_l) \neq S(k)$ for all $k \neq i_l,\ l = 1, 2, \ldots, r$.

2. If $S(k) > S(i_l)$ for some k, then the arrow connecting vertices i_l and k is directed from k to i_l.

3. If $S(k) < S(i_l)$, then the arrow has a direction from i_l to k.

4. $S(i_1) = S(i_2) = \ldots = S(i_r)$.

It is clear that for $r = 1$ the tournament becomes an ordinary *strange tournament*, and such tournaments appear when $r = 2l - 1$, $l \in N$.

Lemma 1. 1. *Let $G = (V, E)$ be a complete (undirected) graph with $|V| = 2s + 1$, $s \geqslant 1$. The graph G can be exchanged to directed graph $\vec{G}$ such that the number of entering and leaving arcs of any vertex of $\vec{G}$ are the same.*

2. *Let $G = (V, E)$ be a complete (undirected) graph with $|V| = 2s$, $s \geqslant 1$. The graph G can be exchanged to directed graph $\vec{G}$ such that the difference of the number of entering and the number leaving arcs of any vertex of $\vec{G}$ less than or equal to* 1.

◁ (1) We use method of induction for vertices of the graph $\vec{G}$. If $|V| = 3$ then it will be a cycle in directed graph. Suppose that for $|V| = 2s + 1$ the Lemma holds, then we shall prove the Lemma for the case $|V| = 2s + 3$. Let $V = \{1, 2, 3, \dots, 2s + 1, 2s + 2, 2s + 3\}$ and $V_1 = \{1, 2, 3, \dots, 2s + 1\}$. By induction we obtain directed graph $\vec{G}_1 = \{V_1, E_1\}$ which the number of entering and leaving arcs of any vertex of $\vec{G}_1$ are the same. Now we add the remaining vertices $2s + 2$, $2s + 3$ as follows:

$$\begin{cases} x_{2s+2} \to x_i, & i \in \{1, 3, 5, \dots, 2s + 1\}, \\ x_{2s+2} \leftarrow x_i, & i \in \{2, 4, 6, \dots, 2s\}. \end{cases}$$

Similarly,

$$\begin{cases} x_{2s+3} \leftarrow x_i, & i \in \{1, 3, 5, \dots, 2s + 1\}, \\ x_{2s+3} \to x_i, & i \in \{2, 4, 6, \dots, 2s + 2\}. \end{cases}$$

Hence, we obtain directed graph $\vec{G} = (V, E)$ such that the number of entering and leaving arcs of any vertex of $\vec{G}$ are the same.

(2) For proving the second part of we use the first part of the Lemma. Let $V = \{1, 2, 3, \dots, 2s + 1, 2s + 2\}$ and $V_1 = \{1, 2, 3, \dots, 2s + 1\}$. By the first part of Lemma we obtain directed graph $\vec{G}_1 = \{V_1, E_1\}$ which the number of entering and leaving arcs of any vertex of $\vec{G}_1$ are the same. We add the remaining vertex $2s + 2$ as follows:

$$\begin{cases} x_{2s+2} \to x_i, & i \in \{1, 3, 5, \dots, 2s + 1\}, \\ x_{2s+2} \leftarrow x_i, & i \in \{2, 4, 6, \dots, 2s\}. \end{cases}$$

Then $S(x_i) = s + 1$ for all $i \in \{1, 3, 5, \dots, 2s + 1\}$ and $S(x_i) = s$ for all $i \in \{2, 4, 6, \dots, 2s, 2s + 2\}$. ▷

Theorem 1. *The following statements hold:*

1. *Let T_m be a strange tournament with vertices $\{1, 2, 3, \dots, m\}$. If $k \in \{1, 2, 3, \dots, m\}$ is a strange vertex, then $\frac{m+5}{3} \leqslant k \leqslant \frac{2m-3}{3}$, $m \geqslant 7$.*

2. *Let $\frac{m+5}{3} \leqslant k \leqslant \frac{2m-3}{3}$, $m \geqslant 7$. Then there exists a strange tournament T_m with vertices $\{1, 2, 3, \dots, m\}$ such that k is a strange vertex.*

◁ (1) In the tournament there are $k - 1$ vertices (without loss of generality $\{1, 2, 3, \dots, k - 1\}$) such that $S(k) < S(i)$ for all $i \in \{1, 2, 3, \dots, k - 1\}$. Also, we have $S(k) > S(j)$ for all $j \in \{k + 1, k + 2, \dots, m\}$. The number of all arcs among the vertices $\{1, 2, 3, \dots, k - 1\}$ is equal to $\binom{k-1}{2}$. The number of arcs entering to the vertex $i \in \{1, 2, 3, \dots, k - 1\}$ from the vertices: $k + 1, k + 2, \dots, m$ less than or equal to $m - k$. Then total number of arcs entering to the vertices: $1, 2, 3, \dots, k - 1$ is at most $\binom{k-1}{2} + (k - 1)(m - k)$. Hence from k is a strange vertex one gets

$$\binom{k-1}{2} + (k - 1)(m - k) > k(k - 1).$$

By pigeonhole principle, we rewrite the last inequality as follows:

$$\binom{k-1}{2} + (k - 1)(m - k) \geqslant (k - 1)^2. \tag{2.1}$$

Inequality (2.1) is equivalent to $k < \frac{2m-3}{3}$. Also, the number of all arcs among the vertices $\{k+1, k+2, \ldots, m\}$ is equal to $\binom{m-k}{2}$. The number of arcs from strange vertex to the vertices: $k+1, k+2, \ldots, m$ is equal to $m-k$. Then one gets:

$$\binom{m-k}{2} + m - k < k(m-k).$$

Again we use pigeonhole principle and rewrite the last inequality as follows:

$$\binom{m-k}{2} + m - k \leqslant (k-1)(m-k). \tag{2.2}$$

From (2.2), we have $k \geqslant \frac{m+5}{3}$.

(2) Let $\frac{m+5}{3} \leqslant k \leqslant \frac{2m-3}{3}$, $m \geqslant 7$. For a fixed k we construct strange tournament. If $m-k$ is even number (the case even is similar), then by the first part of Lemma 1 we can show sub-tournament T' with vertices $k+1, k+2, \ldots, m$ such that the number of entering and leaving arcs of any vertex of T' are the same. Indeed, by Lemma 1 we can construct directed subgraph with $S(i) = S(j)$ for all $i, j \in \{k+1, k+2, \ldots, m\}$. From the inequality $k \leqslant \frac{2m-3}{3}$ we have $S(k) > S(k+1)$. Analogously, since the second part of Lemma 1 and $k \geqslant \frac{m+5}{3}$ we obtain $S(k) < S(i)$, for all $i \in \{1, 2, \ldots, k-1\}$. $\triangleright$

Corollary 1. *There is not any strange tournament among the tournaments T_8. Indeed, there is not any integer k with $\frac{13}{3} \leqslant k \leqslant \frac{15}{3}$. In the case of T_7, by Theorem* 1 (*i. e.*, $4 \leqslant k \leqslant \frac{14}{3}$) *there is a unique* (*up to permutation of vertices*) *strange tournament. Since $k = 4$ and Lemma* 1 *we can conclude this strange tournament as follows:* (4443222) (*see Fig.* 1).

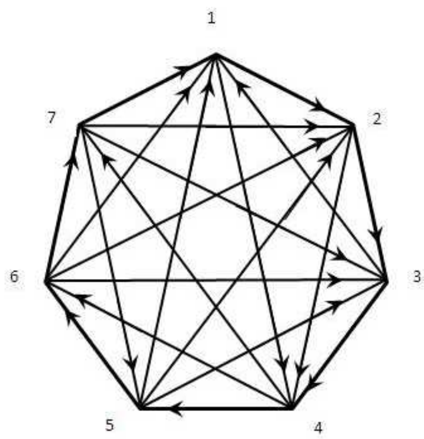

Fig. 1.

In genetics, Q.S.O.V.T. V has the following interpretation:

If species 1, 2, 3 dominate in a population under panmixia, and species 5, 6, 7 are on the verge of extinction, then sometimes species 4 can be found, and it ensures the preservation of all species.

Q.S.O.V.T. V corresponding to this *weird* tournament is:

$$\begin{cases} x_1' = x_1(1 - a_{12}x_2 + a_{13}x_3 - a_{14}x_4 + a_{15}x_5 + a_{16}x_6 + a_{17}x_7), \\ x_2' = x_2(1 + a_{12}x_1 - a_{23}x_3 - a_{24}x_4 + a_{25}x_5 + a_{26}x_6 + a_{27}x_7), \\ x_3' = x_3(1 - a_{13}x_1 + a_{23}x_2 - a_{34}x_4 + a_{35}x_5 + a_{36}x_6 + a_{37}x_7), \\ x_4' = x_4(1 + a_{14}x_1 + a_{24}x_2 + a_{34}x_3 - a_{45}x_5 - a_{46}x_6 - a_{47}x_7), \\ x_5' = x_5(1 - a_{15}x_1 - a_{25}x_2 - a_{35}x_3 + a_{45}x_4 - a_{56}x_6 + a_{57}x_7), \\ x_6' = x_6(1 - a_{16}x_1 - a_{26}x_2 - a_{36}x_3 + a_{46}x_4 + a_{56}x_5 - a_{67}x_7), \\ x_7' = x_7(1 - a_{17}x_1 - a_{27}x_2 - a_{37}x_3 + a_{47}x_4 - a_{57}x_5 + a_{67}x_6), \end{cases} \tag{2.3}$$

where $a_{ki} \in [0;1]$ or $a_{ki} \in [-1;0]$ at the same time.

Consider only the case $a_{ki} \in [0;1]$ otherwise the arcs in the tournament get the opposite direction.

Q.S.O.V.T. (3) except vertices

$$M_1 = (1,0,0,0,0,0,0),\ M_2 = (0,1,0,0,0,0,0),\ \ldots,\ M_7 = (0,0,0,0,0,0,1)$$

has on the boundary, corresponding to cyclic triples, fixed points:

$$C_{123},\ C_{145},\ C_{146},\ C_{147},\ C_{245},\ C_{246},\ C_{247},\ C_{345},\ C_{346},\ C_{347},\ C_{567}.$$

For some of coefficients a_{ki}, there can be an internal fixed point:

$$C\left(\frac{\Delta_1}{\Delta}, \frac{\Delta_2}{\Delta}, \frac{\Delta_3}{\Delta}, \frac{\Delta_4}{\Delta}, \frac{\Delta_5}{\Delta}, \frac{\Delta_6}{\Delta}, \frac{\Delta_7}{\Delta}\right),$$

where

$$\begin{aligned} \Delta_1 = {} & a_{67}a_{24}a_{35} + a_{26}a_{47}a_{35} + a_{45}a_{27}a_{36} + a_{36}a_{24}a_{57} + a_{37}a_{24}a_{56} + a_{46}a_{37}a_{25} \\ & + a_{45}a_{67}a_{23} + a_{56}a_{47}a_{25} + a_{46}a_{23}a_{57} - a_{56}a_{27}a_{34} - a_{25}a_{67}a_{34} \\ & - a_{34}a_{26}a_{57} - a_{46}a_{35}a_{27} - a_{25}a_{47}a_{36} - a_{45}a_{26}a_{37}, \\ \Delta_2 = {} & a_{45}a_{67}a_{13} + a_{56}a_{47}a_{13} + a_{46}a_{13}a_{57} + a_{45}a_{16}a_{37} + a_{34}a_{16}a_{57} + a_{15}a_{67}a_{34} \\ & + a_{56}a_{17}a_{34} + a_{46}a_{35}a_{17} + a_{15}a_{47}a_{6} - a_{14}a_{56}a_{37} - a_{46}a_{37}a_{25} \\ & - a_{67}a_{14}a_{35} - a_{16}a_{47}a_{35} - a_{45}a_{17}a_{36} - a_{36}a_{14}a_{57}, \\ \Delta_3 = {} & a_{47}a_{56}a_{12} + a_{47}a_{25}a_{16} + a_{46}a_{12}a_{57} + a_{45}a_{67}a_{12} + a_{25}a_{14}a_{67} + a_{26}a_{14}a_{57} \\ & + a_{45}a_{17}a_{26} + a_{56}a_{14}a_{27} + a_{46}a_{27}a_{15} - a_{47}a_{15}a_{26} - a_{15}a_{67}a_{24} \\ & - a_{24}a_{16}a_{57} - a_{56}a_{17}a_{24} - a_{46}a_{25}a_{17} - a_{45}a_{16}a_{27}, \\ \Delta_4 = {} & a_{56}a_{17}a_{25} + a_{15}a_{67}a_{23} + a_{23}a_{16}a_{57} + a_{67}a_{13}a_{25} + a_{16}a_{37}a_{25} + a_{36}a_{27}a_{15} \\ & + a_{13}a_{56}a_{27} + a_{35}a_{17}a_{26} + a_{26}a_{13}a_{57} + a_{35}a_{67}a_{12} + a_{56}a_{37}a_{12} \\ & + a_{36}a_{12}a_{57} - a_{36}a_{25}a_{17} - a_{35}a_{16}a_{27} - a_{15}a_{37}a_{26}, \end{aligned}$$

$$\Delta_5 = a_{67}a_{34}a_{12} + a_{36}a_{47}a_{12} + a_{23}a_{16}a_{47} + a_{36}a_{24}a_{57} + a_{37}a_{24}a_{56} + a_{46}a_{37}a_{25} + a_{45}a_{67}a_{23} + a_{56}a_{47}a_{25} + a_{46}a_{23}a_{57} - a_{56}a_{27}a_{34} - a_{25}a_{67}a_{34} - a_{34}a_{26}a_{57} - a_{46}a_{35}a_{27} - a_{25}a_{47}a_{36} - a_{45}a_{26}a_{37},$$

$$\Delta_6 = a_{17}a_{35}a_{24} + a_{57}a_{13}a_{24} + a_{14}a_{37}a_{25} + a_{13}a_{45}a_{27} + a_{15}a_{34}a_{27} + a_{17}a_{45}a_{23} + a_{14}a_{57}a_{23} + a_{45}a_{37}a_{12} + a_{34}a_{57}a_{12} - a_{37}a_{15}a_{24} - a_{47}a_{13}a_{25} - a_{17}a_{34}a_{25} - a_{35}a_{14}a_{27} - a_{47}a_{15}a_{23} - a_{47}a_{35}a_{12},$$

$$\Delta_7 = a_{34}a_{56}a_{12} + a_{46}a_{12}a_{35} + a_{35}a_{26}a_{14} + a_{23}a_{15}a_{46} + a_{14}a_{23}a_{56} + a_{15}a_{36}a_{24} + a_{56}a_{13}a_{24} + a_{34}a_{16}a_{25} + a_{25}a_{13}a_{46} - a_{45}a_{36}a_{12} - a_{13}a_{45}a_{26} - a_{34}a_{15}a_{26} - a_{16}a_{45}a_{23} - a_{35}a_{24}a_{16} - a_{14}a_{36}a_{25},$$

$$\Delta = \sum_{i=1}^{7} \Delta_i.$$

At first, we shall construct the functionals

$$\varphi_{ijk} = \left(x_i^{a_{jk}} \cdot x_j^{a_{ik}} \cdot x_k^{a_{ij}}\right)^{\frac{1}{\Delta_{ijk}}}$$

with respect to the fixed points of C_{ijk} and estimate them by using Young's inequality [12].

$$\varphi_{145}(x') \leqslant \frac{\varphi_{145}(x)}{\Delta_{145}}\big[\Delta_{145} - (a_{12}a_{45} + a_{14}a_{25} - a_{15}a_{24})x_2 + (a_{13}a_{45} + a_{15}a_{34} - a_{14}a_{35})x_3 - (a_{15}a_{46} + a_{14}a_{56} - a_{16}a_{45})x_6 + (a_{17}a_{45} + a_{14}a_{57} - a_{15}a_{47})x_7\big],$$

$$\varphi_{146}(x') \leqslant \frac{\varphi_{146}(x)}{\Delta_{146}}\big[\Delta_{146} - (a_{12}a_{46} + a_{26}a_{14} - a_{16}a_{24})x_2 + (a_{13}a_{46} + a_{16}a_{34} - a_{14}a_{36})x_3 + (a_{46}a_{15} + a_{56}a_{14} - a_{16}a_{45})x_5 - (a_{47}a_{16} + a_{67}a_{14} - a_{17}a_{46})x_7\big],$$

$$\varphi_{147}(x') \leqslant \frac{\varphi_{147}(x)}{\Delta_{147}}\big[\Delta_{147} - (a_{12}a_{47} + a_{27}a_{14} - a_{24}a_{17})x_2 + (a_{13}a_{47} + a_{34}a_{17} - a_{37}a_{14})x_3 - (a_{45}a_{17} + a_{57}a_{14} - a_{15}a_{47})x_5 + (a_{16}a_{47} + a_{67}a_{14} - a_{46}a_{17})x_6\big],$$

$$\varphi_{245}(x') \leqslant \frac{\varphi_{245}(x)}{\Delta_{245}}\big[\Delta_{245} + (a_{12}a_{45} + a_{14}a_{25} - a_{15}a_{24})x_1 - (a_{23}a_{45} + a_{35}a_{24} - a_{34}a_{25})x_3 - (a_{46}a_{25} + a_{56}a_{24} - a_{26}a_{45})x_6 + (a_{27}a_{45} + a_{57}a_{24} - a_{47}a_{25})x_7\big],$$

$$\varphi_{246}(x') \leqslant \frac{\varphi_{246}(x)}{\Delta_{246}}\big[\Delta_{246} + (a_{12}a_{46} + a_{14}a_{26} - a_{16}a_{24})x_2 - (a_{23}a_{46} + a_{36}a_{24} - a_{34}a_{26})x_3 + (a_{25}a_{46} + a_{56}a_{24} - a_{45}a_{26})x_5 - (a_{47}a_{26} + a_{67}a_{24} - a_{27}a_{46})x_7\big],$$

$$\varphi_{247}(x') \leqslant \frac{\varphi_{247}(x)}{\Delta_{247}}\big[\Delta_{247} + (a_{12}a_{47} + a_{14}a_{27} - a_{17}a_{24})x_1 - (a_{23}a_{47} + a_{37}a_{24} - a_{34}a_{27})x_3 - (a_{45}a_{27} + a_{57}a_{24} - a_{25}a_{47})x_5 + (a_{26}a_{47} + a_{24}a_{67} - a_{46}a_{27})x_6\big],$$

$$\varphi_{345}(x') \leqslant \frac{\varphi_{345}(x)}{\Delta_{345}}\big[\Delta_{345} - (a_{13}a_{45} + a_{15}a_{34} - a_{14}a_{35})x_1 + (a_{23}a_{45} + a_{24}a_{35} - a_{25}a_{34})x_2 - (a_{46}a_{35} + a_{56}a_{34} - a_{36}a_{45})x_6 + (a_{37}a_{45} + a_{57}a_{34} - a_{47}a_{35})x_7\big],$$

$$\varphi_{346}(x') \leqslant \frac{\varphi_{346}(x)}{\Delta_{346}}\big[\Delta_{346} - (a_{13}a_{46} + a_{16}a_{34} - a_{14}a_{36})x_1 + (a_{23}a_{46} + a_{24}a_{36} - a_{26}a_{34})x_2 + (a_{46}a_{35} + a_{56}a_{34} - a_{35}a_{46})x_5 - (a_{47}a_{36} + a_{67}a_{34} - a_{37}a_{46})x_7\big],$$

$$\varphi_{347}(x') \leqslant \frac{\varphi_{347}(x)}{\Delta_{347}} \big[\Delta_{347} - (a_{13}a_{47} + a_{17}a_{34} - a_{14}a_{57})x_1 + (a_{23}a_{47} + a_{24}a_{37} - a_{27}a_{34})x_2 - (a_{45}a_{37} + a_{57}a_{34} - a_{35}a_{47})x_5 + (a_{36}a_{47} + a_{67}a_{34} - a_{46}a_{37})x_6\big],$$

$$\varphi_{123}(x') \leqslant \frac{\varphi_{123}(x)}{\Delta_{123}} \big[\Delta_{123} - (a_{14}a_{23} + a_{24}a_{13} + a_{34}a_{12})x_4 + (a_{15}a_{23} + a_{25}a_{13} - a_{35}a_{12})x_5 + (a_{16}a_{25} + a_{26}a_{13} + a_{36}a_{12})x_6 + (a_{17}a_{23} + a_{27}a_{13} + a_{37}a_{12})x_7\big],$$

$$\varphi_{567}(x') \leqslant \frac{\varphi_{567}(x)}{\Delta_{567}} \big[\Delta_{567} - (a_{15}a_{67} + a_{16}a_{57} + a_{17}a_{56})x_1 - (a_{25}a_{67} + a_{26}a_{57} + a_{27}a_{56})x_2 - (a_{35}a_{67} + a_{36}a_{57} + a_{37}a_{56})x_3 + (a_{45}a_{67} + a_{46}a_{57} + a_{47}a_{56})x_4\big].$$

Denoting the inner brackets, we have

$$\varphi_{145}(x') \leqslant \frac{\varphi_{145}(x)}{\Delta_{145}} \big[\Delta_{145} - K_1x_2 + K_2x_3 - K_3x_6 + K_4x_7\big],$$

$$\varphi_{146}(x') \leqslant \frac{\varphi_{146}(x)}{\Delta_{146}} \big[\Delta_{146} - K_5x_2 + K_6x_3 + K_3x_5 - K_7x_7\big],$$

$$\varphi_{147}(x') \leqslant \frac{\varphi_{147}(x)}{\Delta_{147}} \big[\Delta_{147} - K_8x_2 + K_9x_3 - K_4x_5 + K_7x_6\big],$$

$$\varphi_{245}(x') \leqslant \frac{\varphi_{245}(x)}{\Delta_{245}} \big[\Delta_{245} + K_1x_1 - K_{10}x_3 - K_4x_6 + K_{12}x_7\big],$$

$$\varphi_{246}(x') \leqslant \frac{\varphi_{246}(x)}{\Delta_{246}} \big[\Delta_{246} - K_5x_1 - K_{13}x_3 + K_{11}x_5 - K_{14}x_7\big],$$

$$\varphi_{247}(x') \leqslant \frac{\varphi_{247}(x)}{\Delta_{247}} \big[\Delta_{247} - K_8x_1 - K_{15}x_3 - K_{12}x_5 + K_{14}x_6\big],$$

$$\varphi_{345}(x') \leqslant \frac{\varphi_{345}(x)}{\Delta_{345}} \big[\Delta_{345} - K_2x_1 + K_{10}x_2 - K_{16}x_6 + K_{17}x_7\big],$$

$$\varphi_{346}(x') \leqslant \frac{\varphi_{346}(x)}{\Delta_{346}} \big[\Delta_{346} - K_6x_1 + K_{13}x_2 + K_{16}x_5 - K_{18}x_7\big],$$

$$\varphi_{347}(x') \leqslant \frac{\varphi_{347}(x)}{\Delta_{347}} \big[\Delta_{347} - K_9x_1 + K_{15}x_2 - K_{17}x_5 + K_{18}x_6\big],$$

$$\varphi_{123}(x') \leqslant \frac{\varphi_{123}(x)}{\Delta_{123}} \big[\Delta_{123} - K_{19}x_4 + K_{20}x_5 + K_{21}x_6 + K_{22}x_7\big],$$

$$\varphi_{567}(x') \leqslant \frac{\varphi_{567}(x)}{\Delta_{567}} \big[\Delta_{567} - K_{23}x_1 - K_{24}x_2 - K_{25}x_3 + K_{26}x_4\big].$$

If there is no Lyapunov function among the functionals $\varphi_{ijk}(x)$, then there exists an interior fixed point C. We will study only such cases.

Lemma 2. *Let $x \in \operatorname{int} S^6$, $x \neq C$, then the limit set of trajectories is infinite and lies on the boundary of the simplex S^6, i. e., $\omega(x) \subset \partial S^6$.*

◁ Now we consider the following functional

$$\varphi(x) = \left(\prod_{k=1}^{7} x_k^{\Delta_k}\right)^{\frac{1}{\Delta}}.$$

Because

$$\varphi(x') = \varphi(x)\big[1 - a_{12}x_2 + a_{13}x_3 - a_{14}x_4 + a_{15}x_5 + a_{16}x_6 + a_{17}x_7\big]^{\frac{\Delta_1}{\Delta}} \big[1 + a_{12}x_1 - a_{23}x_3 - a_{24}x_4 + a_{25}x_5 + a_{26}x_6 + a_{27}x_7\big]^{\frac{\Delta_2}{\Delta}} \big[1 - a_{13}x_1 - a_{23}x_2 - a_{34}x_4 + a_{35}x_5 + a_{26}x_6 + a_{27}x_7\big]^{\frac{\Delta_3}{\Delta}} \big[1 - a_{14}x_1 - a_{24}x_2 + a_{34}x_3 - a_{45}x_5 + a_{26}x_6 + a_{27}x_7\big]^{\frac{\Delta_4}{\Delta}} \big[1 - a_{15}x_1 - a_{25}x_2 - a_{35}x_3 + a_{45}x_4 + a_{26}x_6 + a_{27}x_7\big]^{\frac{\Delta_5}{\Delta}} \big[1 - a_{16}x_1 - a_{26}x_2 - a_{36}x_3 + a_{46}x_4 + a_{56}x_5 - a_{67}x_7\big]^{\frac{\Delta_6}{\Delta}} \big[1 - a_{17}x_1 - a_{27}x_2 - a_{37}x_3 + a_{47}x_4 - a_{57}x_5 + a_{67}x_6\big]^{\frac{\Delta_7}{\Delta}}$$
$$\leqslant \frac{\varphi(x)}{\Delta}\Big[\Delta_1(1 - a_{12}x_2 + a_{13}x_3 - a_{14}x_4 + a_{15}x_5 + a_{16}x_6 + a_{17}x_7) + \Delta_2(1 + a_{12}x_1 - a_{23}x_3 - a_{24}x_4 + a_{25}x_5 + a_{26}x_6 + a_{27}x_7) + \Delta_3(1 - a_{13}x_1 - a_{23}x_2 - a_{34}x_4 + a_{35}x_5 + a_{26}x_6 + a_{27}x_7) + \Delta_4(1 - a_{14}x_1 - a_{24}x_2 + a_{34}x_3 - a_{45}x_5 + a_{26}x_6 + a_{27}x_7) + \Delta_5(1 - a_{15}x_1 - a_{25}x_2 - a_{35}x_3 + a_{45}x_4 + a_{26}x_6 + a_{27}x_7) + \Delta_6(1 - a_{16}x_1 - a_{26}x_2 - a_{36}x_3 + a_{46}x_4 + a_{56}x_5 - a_{67}x_7) + \Delta_7(1 - a_{17}x_1 - a_{27}x_2 - a_{37}x_3 + a_{47}x_4 - a_{57}x_5 + a_{67}x_6)\Big] = \varphi(x),$$

$\varphi(V^n x)$ decreases as $n \to \infty$, i. e.,

$$\lim_{n\to\infty} \varphi(V^n x) = 0.$$

Hence $\omega(x) \subset \partial S^6$. $\rhd$

From the invariance of fixed vertices, edges and faces of S^6, the limit set cannot be finite. Thus, (2.3) can be rewritten as following form:

$$\begin{cases} x_1' = x_1[1 - (a_{12}x_2 + a_{14}x_4) + (a_{13}x_3 + a_{15}x_5 + a_{16}x_6 + a_{17}x_7)] = x_1[1 - \Delta_{24} + \Delta_{3567}], \\ x_2' = x_2[1 - (a_{23}x_3 + a_{24}x_4) + (a_{12}x_1 + a_{25}x_5 + a_{26}x_6 + a_{27}x_7)] = x_2[1 - \Delta_{34} + \Delta_{1567}], \\ x_3' = x_3[1 - (a_{13}x_1 + a_{34}x_4) + (a_{23}x_2 + a_{35}x_5 + a_{26}x_6 + a_{27}x_7)] = x_3[1 - \Delta_{14} + \Delta_{2567}], \\ x_4' = x_4[1 - (a_{14}x_1 + a_{24}x_2 + a_{45}x_5) + (a_{34}x_3 + a_{26}x_6 + a_{27}x_7)] = x_4[1 - \Delta_{123} + \Delta_{567}], \\ x_5' = x_5[1 - (a_{15}x_1 + a_{25}x_2 + a_{35}x_3 + a_{56}x_6) + (a_{45}x_4 + a_{57}x_7)] = x_5[1 - \Delta_{1236} + \Delta_{47}], \\ x_6' = x_6[1 - (a_{16}x_1 + a_{26}x_2 + a_{36}x_3 + a_{67}x_7) + (a_{46}x_4 + a_{56}x_5)] = x_6[1 - \Delta_{1237} + \Delta_{45}], \\ x_7' = x_7[1 - (a_{17}x_1 + a_{27}x_2 + a_{37}x_3 + a_{57}x_5) + (a_{47}x_4 + a_{67}x_6)] = x_7[1 - \Delta_{1235} + \Delta_{46}]. \end{cases}$$

Let us split the S^6 simplex into the following parts:

$$T_1 = \big\{[\Delta_{24} \leqslant \Delta_{3567}] \cap [\Delta_{34} \leqslant \Delta_{1567}] \cap [\Delta_{14} \leqslant \Delta_{2567}] \cap [\Delta_{567} \leqslant \Delta_{123}] \cap [\Delta_{47} \leqslant \Delta_{1236}] \cap [\Delta_{45} \leqslant \Delta_{1237}] \cap [\Delta_{46} \leqslant \Delta_{1235}]\big\},$$
$$T_2 = \big\{[\Delta_{24} \geqslant \Delta_{3567}] \cap [\Delta_{34} \leqslant \Delta_{1567}] \cap [\Delta_{14} \leqslant \Delta_{2567}] \cap [\Delta_{567} \leqslant \Delta_{123}] \cap [\Delta_{47} \leqslant \Delta_{1236}] \cap [\Delta_{45} \leqslant \Delta_{1237}] \cap [\Delta_{46} \leqslant \Delta_{1235}]\big\},$$
$$T_3 = \big\{[\Delta_{24} \geqslant \Delta_{3567}] \cap [\Delta_{34} \leqslant \Delta_{1567}] \cap [\Delta_{14} \leqslant \Delta_{2567}] \cap [\Delta_{567} \leqslant \Delta_{123}] \cap [\Delta_{47} \geqslant \Delta_{1236}] \cap [\Delta_{45} \leqslant \Delta_{1237}] \cap [\Delta_{46} \leqslant \Delta_{1235}]\big\},$$

$$T_4 = \big\{[\Delta_{24} \geqslant \Delta_{3567}] \cap [\Delta_{34} \geqslant \Delta_{1567}] \cap [\Delta_{14} \leqslant \Delta_{2567}] \cap [\Delta_{567} \leqslant \Delta_{123}] \cap \\ \cap [\Delta_{47} \geqslant \Delta_{1236}] \cap [\Delta_{45} \leqslant \Delta_{1237}] \cap [\Delta_{46} \leqslant \Delta_{1235}]\big\},$$

$$T_5 = \big\{[\Delta_{24} \geqslant \Delta_{3567}] \cap [\Delta_{34} \geqslant \Delta_{1567}] \cap [\Delta_{14} \leqslant \Delta_{2567}] \cap [\Delta_{567} \leqslant \Delta_{123}] \\ \cap [\Delta_{47} \geqslant \Delta_{1236}] \cap [\Delta_{45} \geqslant \Delta_{1237}] \cap [\Delta_{46} \leqslant \Delta_{1235}]\big\},$$

$$T_6 = \big\{[\Delta_{24} \geqslant \Delta_{3567}] \cap [\Delta_{34} \geqslant \Delta_{1567}] \cap [\Delta_{14} \geqslant \Delta_{2567}] \cap [\Delta_{567} \leqslant \Delta_{123}] \\ \cap [\Delta_{47} \geqslant \Delta_{1236}] \cap [\Delta_{45} \geqslant \Delta_{1237}] \cap [\Delta_{46} \leqslant \Delta_{1235}]\big\},$$

$$T_7 = \big\{[\Delta_{24} \geqslant \Delta_{3567}] \cap [\Delta_{34} \geqslant \Delta_{1567}] \cap [\Delta_{14} \geqslant \Delta_{2567}] \cap [\Delta_{567} \leqslant \Delta_{123}] \\ \cap [\Delta_{47} \geqslant \Delta_{1236}] \cap [\Delta_{45} \geqslant \Delta_{1237}] \cap [\Delta_{46} \geqslant \Delta_{1235}]\big\},$$

$$T_8 = \big\{[\Delta_{24} \geqslant \Delta_{3567}] \cap [\Delta_{34} \geqslant \Delta_{1567}] \cap [\Delta_{14} \geqslant \Delta_{2567}] \cap [\Delta_{567} \geqslant \Delta_{123}] \\ \cap [\Delta_{47} \geqslant \Delta_{1236}] \cap [\Delta_{45} \geqslant \Delta_{1237}] \cap [\Delta_{46} \geqslant \Delta_{1235}]\big\},$$

$$T_9 = \big\{[\Delta_{24} \leqslant \Delta_{3567}] \cap [\Delta_{34} \geqslant \Delta_{1567}] \cap [\Delta_{14} \geqslant \Delta_{2567}] \cap [\Delta_{567} \geqslant \Delta_{123}] \\ \cap [\Delta_{47} \geqslant \Delta_{1236}] \cap [\Delta_{45} \geqslant \Delta_{1237}] \cap [\Delta_{46} \geqslant \Delta_{1235}]\big\},$$

$$T_{10} = \big\{[\Delta_{24} \leqslant \Delta_{3567}] \cap [\Delta_{34} \geqslant \Delta_{1567}] \cap [\Delta_{14} \geqslant \Delta_{2567}] \cap [\Delta_{567} \geqslant \Delta_{123}] \\ \cap [\Delta_{47} \leqslant \Delta_{1236}] \cap [\Delta_{45} \geqslant \Delta_{1237}] \cap [\Delta_{46} \geqslant \Delta_{1235}]\big\},$$

$$T_{11} = \big\{[\Delta_{24} \leqslant \Delta_{3567}] \cap [\Delta_{34} \leqslant \Delta_{1567}] \cap [\Delta_{14} \geqslant \Delta_{2567}] \cap [\Delta_{567} \geqslant \Delta_{123}] \\ \cap [\Delta_{47} \leqslant \Delta_{1236}] \cap [\Delta_{45} \geqslant \Delta_{1237}] \cap [\Delta_{46} \geqslant \Delta_{1235}]\big\},$$

$$T_{12} = \big\{[\Delta_{24} \leqslant \Delta_{3567}] \cap [\Delta_{34} \leqslant \Delta_{1567}] \cap [\Delta_{14} \geqslant \Delta_{2567}] \cap [\Delta_{567} \geqslant \Delta_{123}] \\ \cap [\Delta_{47} \leqslant \Delta_{1236}] \cap [\Delta_{45} \leqslant \Delta_{1237}] \cap [\Delta_{46} \geqslant \Delta_{1235}]\big\},$$

$$T_{13} = \big\{[\Delta_{24} \leqslant \Delta_{3567}] \cap [\Delta_{34} \leqslant \Delta_{1567}] \cap [\Delta_{14} \leqslant \Delta_{2567}] \cap [\Delta_{567} \leqslant \Delta_{123}] \\ \cap [\Delta_{47} \leqslant \Delta_{1236}] \cap [\Delta_{45} \leqslant \Delta_{1237}] \cap [\Delta_{46} \leqslant \Delta_{1235}]\big\},$$

$$T_{14} = \big\{[\Delta_{24} \leqslant \Delta_{3567}] \cap [\Delta_{34} \leqslant \Delta_{1567}] \cap [\Delta_{14} \leqslant \Delta_{2567}] \cap [\Delta_{567} \geqslant \Delta_{123}] \\ \cap [\Delta_{47} \leqslant \Delta_{1236}] \cap [\Delta_{45} \leqslant \Delta_{1237}] \cap [\Delta_{46} \leqslant \Delta_{1235}]\big\}.$$

Lemma 3. *For any point $x \in \mathrm{int}\, S^6$, $x \neq C$, its route is given by the diagram*

$$T_1 \to T_2 \to T_3 \to \ldots \to T_{14} \to T_1.$$

$\lhd$ Let $x \in T_1$. Then

$$x_1' > x_1,\ x_2' > x_2,\ x_3' > x_3,\ x_4' > x_4,\ x_5' > x_5,\ x_6' > x_6,\ x_7' > x_7$$

and after some time the trajectory hits T_2. The other cases can be verified similarly. It is clear that

$$\frac{\Delta_j}{\Delta}\, x_i = \frac{\Delta_i}{\Delta}\, x_j.$$

Close to the vertices M_i $(i = \overline{1,7}\,)$

$$\frac{\Delta_j}{\Delta}\, x_i \geqslant \frac{\Delta_i}{\Delta}\, x_j,$$

which will be denoted by $x_i \succ x_j$. Then the partition of the S^6 simplex can be carried out as follows:

$$G_1 = \{x \in S^6 : x_1 \succ x_7 \succ x_2 \succ x_6 \succ x_3 \succ x_5 \succ x_4\},$$
$$G_2 = \{x \in S^6 : x_1 \succ x_2 \succ x_7 \succ x_3 \succ x_6 \succ x_4 \succ x_5\},$$
$$G_3 = \{x \in S^6 : x_2 \succ x_1 \succ x_3 \succ x_7 \succ x_4 \succ x_6 \succ x_5\},$$
$$G_4 = \{x \in S^6 : x_2 \succ x_3 \succ x_1 \succ x_4 \succ x_7 \succ x_5 \succ x_6\},$$
$$G_5 = \{x \in S^6 : x_3 \succ x_2 \succ x_4 \succ x_1 \succ x_5 \succ x_7 \succ x_6\},$$
$$G_6 = \{x \in S^6 : x_3 \succ x_4 \succ x_2 \succ x_5 \succ x_1 \succ x_6 \succ x_7\},$$
$$G_7 = \{x \in S^6 : x_4 \succ x_3 \succ x_5 \succ x_2 \succ x_6 \succ x_1 \succ x_7\},$$
$$G_8 = \{x \in S^6 : x_4 \succ x_5 \succ x_3 \succ x_6 \succ x_2 \succ x_7 \succ x_1\},$$
$$G_9 = \{x \in S^6 : x_5 \succ x_4 \succ x_6 \succ x_3 \succ x_7 \succ x_2 \succ x_1\},$$
$$G_{10} = \{x \in S^6 : x_5 \succ x_6 \succ x_4 \succ x_7 \succ x_3 \succ x_1 \succ x_2\},$$
$$G_{11} = \{x \in S^6 : x_6 \succ x_5 \succ x_7 \succ x_4 \succ x_1 \succ x_3 \succ x_2\},$$
$$G_{12} = \{x \in S^6 : x_6 \succ x_7 \succ x_5 \succ x_1 \succ x_4 \succ x_2 \succ x_3\},$$
$$G_{13} = \{x \in S^6 : x_7 \succ x_6 \succ x_1 \succ x_5 \succ x_2 \succ x_4 \succ x_3\},$$
$$G_{14} = \{x \in S^6 : x_7 \succ x_1 \succ x_6 \succ x_2 \succ x_5 \succ x_3 \succ x_4\}.$$

Put

$$H_1 = G_1 \cup G_2,\ H_2 = G_3 \cup G_4,\ H_3 = G_5 \cup G_6,\ H_4 = G_7 \cup G_8,$$
$$H_5 = G_9 \cup G_{10},\ H_6 = G_{11} \cup G_{12},\ H_7 = G_{13} \cup G_{14}.$$

Choose a neighborhood U_0 of C, $U_0 \subset \operatorname{int} S^6$ and $U_i = H_i \backslash U_0$ $(i = \overline{1,7})$ so that they are convex and $\bigcap_{i=1}^{7} U_i = \varnothing$. These partitions T_i and G_i are almost the same. For example, in the partition T_i, the hyperplane $\Delta_{24} = \Delta_{3567}$, which defines the boundaries of T_1, passes through the points M_1, C and

$$C_{45}\left(0,\ 0,\ 0,\ \frac{a_{15}}{a_{14}+a_{15}},\ \frac{a_{14}}{a_{14}+a_{15}},\ 0,\ 0\right),$$

and in the partition G_i the system of hyperplane inequalities T_1 gives a similar part of the simplex whose boundary passes through M_1, C and

$$C'_{45}\left(0,\ 0,\ 0,\ \frac{\Delta_{15}}{\Delta_{14}+\Delta_{15}},\ \frac{\Delta_{14}}{\Delta_{14}+\Delta_{15}},\ 0,\ 0\right).$$

We have changed these partitions in order to easily find the ratio of coordinates near the M_i vertices. Still, the number of iteration steps in both divisions is almost the same. $\triangleright$

Lemma 4. *Let $x \notin U$, $V^k x \in U$ for all $k = \overline{1,n}$ and $V^{n+1} \notin U$, where U is one of the regions U_i, $x \in \operatorname{int} S^6$ and $x \neq C$. Then*

$$n > A \log_2 \frac{B}{\varphi(x)},$$

where A, B are absolute constants.

$\lhd$ Let, for example, $U = U_1$. Then $x \notin U_1$, $x \in G_{14}$, i. e.,

$$x_7 \succ x_1 \succ x_6 \succ x_2 \succ x_5 \succ x_3 \succ x_4.$$

Since $x' = Vx \in U_1$, is precisely $x' \in G_1$, then

$$x_1' \succ x_7' \succ x_2' \succ x_6' \succ x_3' \succ x_5' \succ x_4'.$$

Whence $x_1' \geqslant \frac{\min \Delta_i}{\Delta} = \alpha_1$.

$$\frac{x_7'}{x_1'} \geqslant \min_{x \in G_{14}} \frac{x_7'}{x_1'} = \alpha_7, \quad \frac{x_6'}{x_1'} \geqslant \min_{x \in G_{14}} \frac{x_6'}{x_1'} = \alpha_6, \quad \frac{x_3'}{x_1'} \geqslant \min_{x \in G_{14}} \frac{x_3'}{x_1'} = \alpha_3,$$
$$\frac{x_5'}{x_1'} \geqslant \min_{x \in G_{14}} \frac{x_5'}{x_1'} = \alpha_5, \quad \frac{x_4'}{x_1'} \geqslant \min_{x \in G_{14}} \frac{x_4'}{x_1'} = \alpha_4,$$

where we take min in the region $G_{14} \subset U_7$

Since the interior of the simplex is invariant, all coefficients of α_i are positive. For example, if $\alpha_7 = 0$, then there exists $x \in G_{14}$ such that $\frac{x_7'}{x_1'} = 0$, i. e.,

$$x_7\big[1 - (a_{17}x_1 + a_{27}x_2 + a_{37}x_3 + a_{57}x_5) + (a_{47}x_4 + a_{67}x_6)\big] = 0.$$

If $x_7 = 0$ then from $x_7 \succ x_1 \succ x_6 \succ x_2 \succ x_5 \succ x_3 \succ x_4$ follows

$$x_1 = x_2 = \ldots = x_6 = 0,$$

which is impossible, but

$$\big[1 - (a_{17}x_1 + a_{27}x_2 + a_{37}x_3 + a_{57}x_5) + (a_{47}x_4 + a_{67}x_6)\big] = 0$$

never done, because it's never done

$$(a_{17}x_1 + a_{27}x_2 + a_{37}x_3 + a_{57}x_5) = 1$$

in G_{14}. Let $V^k x = (x_1^k, x_2^k, \ldots, x_7^k)$.

Since $V^{n+1}x \notin U_1$, then $V^{n+1}x \in G_3$:

$$x_2^{n+1} \succ x_1^{n+1} \succ x_3^{n+1} \succ x_7^{n+1} \succ x_4^{n+1} \succ x_6^{n+1} \succ x_5^{n+1}.$$

Whence $x_2^{n+1} > \frac{\min \Delta_i}{\Delta} = \alpha_1$. Next,

$$\frac{x_2^{n+1}}{x_2'} = \prod_{k=1}^{n} \frac{x_2^{k+1}}{x_2^k} = \prod_{k=1}^{n} \left(1 + a_{12}x_1^k - a_{23}x_3^k - a_{24}x_4^k + a_{25}x_5^k + a_{26}x_6^k + a_{27}x_7^k\right) < 2^n,$$

$$a_{12}x_1^k - a_{23}x_3^k - a_{24}x_4^k + a_{25}x_5^k + a_{26}x_6^k + a_{27}x_7^k < 1.$$

Further,

$$2^n > \frac{x_2^{n+1}}{x_2'} \geqslant \frac{\alpha_1}{x_2'} = \left[\frac{\alpha_1^{\Delta_2}}{(x_2')^{\Delta_2}}\right]^{\frac{1}{\Delta_2}}$$
$$= \left[\frac{\alpha_1^{\Delta_\alpha} \cdot (x_1')^{\Delta_1} \cdot (x_3')^{\Delta_3} \cdot (x_4')^{\Delta_4} \cdot (x_6')^{\Delta_6} \cdot (x_7')^{\Delta_7} \cdot (x_5')^{\Delta_5}}{(x_2')^{\Delta_2} \cdot (x_1')^{\Delta_1} \cdot (x_3')^{\Delta_3} \cdot (x_4')^{\Delta_4} \cdot (x_5')^{\Delta_5} \cdot (x_6')^{\Delta_6} \cdot (x_7')^{\Delta_7}}\right]^{\frac{1}{\Delta_2} \cdot \frac{\Delta}{\Delta}}$$
$$\geqslant \frac{\left[\alpha_1^{\Delta_1 + \Delta_2} \cdot \alpha_3^{\Delta_3} \cdot \alpha_4^{\Delta_4} \cdot \alpha_5^{\Delta_5} \cdot \alpha_6^{\Delta_6} \cdot \alpha_7^{\Delta_7}\right]^{\frac{1}{\Delta_2}}}{[\varphi(Vx)]^{\frac{\Delta}{\Delta_2}}},$$

i. e. $n > \frac{1}{\Delta} \log_2 \frac{1}{\varphi(Vx)}$. $\rhd$

Lemma 5. *Let U be one of the domains U_i, $x \in \operatorname{int} S^6$, $x \neq C$. Let $\{n_i, m_i\}_{i=1}^{\infty}$ be sequences of natural numbers such that $V^{n_i}x \notin U$, $V^{n_i+k}x \in U$ for all $k = \overline{1, m_i}$ and $V^{n_i+m_i+1}x \in U$. Then there exists K such that $m_i > kn_i$.*

$$\triangleleft\ \rho = \max_{\operatorname{int} S^6 \setminus U_0} \Big\{ \big[1 - a_{12}x_2 + a_{13}x_3 - a_{14}x_4 + a_{15}x_5 + a_{16}x_6 + a_{17}x_7\big]^{\Delta_1} \big[1 - a_{12}x_1 - a_{23}x_3 - a_{24}x_4 + a_{25}x_5 + a_{26}x_6 + a_{27}x_7\big]^{\Delta_2} \big[1 - a_{13}x_1 + a_{23}x_3 - a_{34}x_4 + a_{35}x_5 + a_{36}x_6 + a_{37}x_7\big]^{\Delta_3} \times \big[1 + a_{14}x_1 + a_{24}x_3 - a_{14}x_4 + a_{15}x_5 + a_{16}x_6 + a_{17}x_7\big]^{\Delta_1} \big[1 - a_{15}x_1 - a_{25}x_2 - a_5x_3 + a_{45}x_4 - a_{56}x_6 + a_{57}x_7\big]^{\Delta_5} \big[1 - a_{16}x_1 - a_{26}x_2 - a_{36}x_3 + a_{46}x_4 + a_{56}x_6 - a_{67}x_7\big]^{\Delta_6} \times \big[1 - a_{17}x_1 - a_{27}x_2 - a_{37}x_3 + a_{47}x_4 - a_{57}x_5 + a_{67}x_6\big]^{\Delta_7} \Big\} < 1.$$

By Lemma 4

$$m_i > A \log_2 \frac{1}{\varphi(V^{n_i}x)} > A \log_2 \frac{1}{\rho^{n_i}\varphi(x_0)} = \log_2 \frac{A'}{\rho^{n_i}} = \log_2 A' + n_i \log_2 \frac{1}{\rho} = kn_i. \quad \triangleright$$

Theorem 2. *Operator V (defined in (2.3)) is non-ergodic, i. e., for any point $x \in \operatorname{int} S^6$, $x \neq C$, the sequence $\frac{1}{n}\sum_{k=0}^{n-1} V^k x$ has no limit.*

$\triangleleft$ Assume the opposite, i. e., for any point $x \in S^6$ there is a limit

$$\lim_{n\to\infty} \sum_{k=0}^{n-1} V^k x = x^*.$$

Let $x^* \notin U_1$, $\{n_i, m_i\}$ be the same as in Lemma 5. Let $d = \operatorname{dist}(x^*, U_1)$ and

$$\operatorname{dist}\left(\frac{1}{n}\sum_{k=0}^{n-1} V^k x, x^*\right) < \frac{d}{k}$$

for sufficiently large n. Since $m_i > kn_i$ for some $\overline{x} \in \operatorname{int} S^6$, $\overline{x} = \frac{1}{n_i+m_i}\sum_{k=1}^{n_i+m_i} V^k x$

$$\overline{x} = \frac{n_i}{n_i+m_i}\left(\frac{1}{n_i}\sum_{k=1}^{n_i+m_i} V^k x\right) + \frac{m_i}{n_i+m_i}\left(\frac{1}{m_i}\sum_{k=n_i+1}^{n_i+m_i} V^k x\right),$$

where the second term in U_1, must be $\operatorname{dist}(\overline{x}, x^*) < \frac{d}{k}$. But it contradicts to $\operatorname{dist}(\overline{x}, x^*) > \frac{d}{k}$ for large n. $\triangleright$

The first 3-strange tournament occurs among the T_{13} tournaments, the order of which is (7777766655555).

Theorem 3. *Q.S.O.V.T. corresponding r to strange tournaments is non-ergodic. The proof is similar to Theorem 1. In general, the following holds:*

Theorem 4. *V has an interior fixed point, then it is non-ergodic.*

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ДИНАМИКА КВАДРАТИЧНЫХ СТОХАСТИЧЕСКИХ ОПЕРАТОРОВ ТИПА ВОЛЬТЕРРА, СООТВЕТСТВУЮЩИХ СТРАННЫМ ТУРНИРАМ

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Аннотация. Изучая динамику названных операторов на симплексе, уделяя особое внимание наличию внутренней неподвижной точки, мы исследуем условия, при которых операторы проявляют неэргодическое поведение. Посредством строгого анализа и численного моделирования мы показываем, что определенные режимы параметров приводят к неэргодичности, характеризующейся сходимостью начальных распределений к ограниченному подмножеству симплекса. Наши результаты проливают свет на сложную динамику квадратичных стохастических операторов с внутренними неподвижными точками и дают представление о возникновении неэргодического поведения в сложных динамических системах. Кроме того, неэргодичность квадратичных стохастических операторов типа Вольтерра с внутренней неподвижной точкой, определенной в симплексе, вносит дополнительную сложность в и без того сложную динамику таких систем. В этом контексте наличие внутренней неподвижной точки внутри симплекса еще больше усложняет исследование пространства состояний и свойства сходимости оператора. В данной статье мы приводим достаточные и необходимые условия существования странных турниров. Также доказывается неэргодичность квадратичных стохастических операторов типа Вольтерра с внутренней неподвижной точкой, определенных в симплексе.

Ключевые слова: квадратичные стохастические операторы типа Вольтерра, симплекс, странные турниры, функции Ляпунова.

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