Владикавказский математический журнал
2013, Том 15, Выпуск 2, С. 28–35

INCLUSION PROPERTIES FOR CERTAIN SUBCLASSES OF p-VALENT FUNCTIONS ASSOCIATED WITH NEW GENERALIZED DERIVATIVE OPERATOR[1]

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In this paper we introduce several new classes of p-valent functions defined by new generalized derivative operator and investigate various inclusion properties of these classes. Some interesting applications involving classes of integral operators are also considered.

Mathematics Subject Classification (2000): 30C45.

Key words: p-valent functions, derivative operator, integral operator, inclusion properties.

1. Introduction

Let $A(p)$ denote the class of functions of form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \quad (p \in \mathbb{N} = \{1, 2, \ldots\}), \tag{1.1}$$

which are analytic and p-valent in the open unit disk $U = \{z : z \in \mathbb{C}, |z| < 1\}$. A function $f \in A(p)$ is said to be in the class $S_p^*(\alpha)$ of p-valently starlike functions of order α in U if and only if

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \quad (0 \leqslant \alpha < p). \tag{1.2}$$

A function $f \in A(p)$ is said to be in the class $C_p(\alpha)$ of p-valently convex functions of order α if and only if

$$\mathrm{Re}\left\{1 + \frac{zf'(z)}{f(z)}\right\} > \alpha \quad (0 \leqslant \alpha < p). \tag{1.3}$$

It is easy to prove from (1.2) and (1.3) that

$$f(z) \in C_p(\alpha) \Leftrightarrow \frac{z}{p} f'(z) \in S_p^*(\alpha). \tag{1.4}$$

For a function $f \in A(p)$ we say that $f \in K_p(\beta, \alpha)$ if there exists a function $g \in S_p^*(\alpha)$ such that

$$\mathrm{Re}\left\{\frac{zf'(z)}{g(z)}\right\} > \beta \quad (z \in U;\ 0 \leqslant \alpha < p,\ 0 \leqslant \beta). \tag{1.5}$$

Functions in the class $K_p(\beta, \alpha)$ are called p-valently close-to-convex functions of order β type α. We also say that a function $f \in A(p)$ is in the class $K_p^*(\beta, \alpha)$ of p-valently quasi convex functions of order β type α if there exists a function $g \in C_p(\alpha)$ such that

$$\mathrm{Re}\left\{\frac{(zf'(z))'}{g'(z)}\right\} > \beta \quad (0 \leqslant \alpha < p,\ 0 \leqslant \beta). \tag{1.6}$$

[1] The work presented here was supported by UKM-ST-06-FRGS0244-2010.

It follows easily from (1.5) and (1.6) that

$$f(z) \in K_p^*(\beta,\alpha) \Leftrightarrow \frac{z}{p} f'(z) \in K_p(\beta,\alpha). \qquad (1.7)$$

Now we will introduce a new generalized derivative operator $D_{p,\lambda}^n f^{(q)}$ is defined by $D_{p,\lambda}^n f^{(q)} : A(p) \to A(p)$. For each $f \in A(p)$ we have

$$f^{(q)}(z) = \frac{p!}{(p-q)!} z^{p-q} + \sum_{k=p+1}^{\infty} \frac{k!}{(k-q)!} a_k z^{k-q} \quad (q \in \mathbb{N}_0 = \mathbb{N} \cup \{0\},\ p > q). \qquad (1.8)$$

For a function $f \in A(p)$ we define $D_{p,\lambda}^0 f^{(q)}(z) = f^{(q)}(z)$.

$$\begin{aligned} D_{p,\lambda}^1 f^{(q)}(z) &= D f^{(q)}(z) = \frac{1}{p+\lambda-q}\Big[z(f^{(q)}(z))' + \lambda f^{(q)}(z)\Big] \\ &= \frac{1}{p+\lambda-q}\Big[z f^{(q+1)}(z) + \lambda f^{(q)}(z)\Big] = \frac{p!}{(p-q)!} z^{p-q} \\ &+ \sum_{k=p+1}^{\infty} \frac{k!}{(k-q)!}\left(\frac{k+\lambda-q}{p+\lambda-q}\right) a_k z^{k-q} \quad (q \in \mathbb{N}_0 = \mathbb{N} \cup \{0\},\ \lambda \geqslant 0,\ p > q). \end{aligned} \qquad (1.9)$$

And

$$\begin{aligned} D_{p,\lambda}^2 f^{(q)}(z) &= D(D_{p,\lambda}^1 f^{(q)}(z)) \\ &= \frac{p!}{(p-q)!} z^{p-q} + \sum_{k=p+1}^{\infty} \frac{k!}{(k-q)!} \left(\frac{k+\lambda-q}{p+\lambda-q}\right)^2 a_k z^{k-q} \\ &(q \in \mathbb{N}_0 = \mathbb{N} \cup \{0\},\ \lambda \geqslant 0,\ p > q). \end{aligned} \qquad (1.10)$$

And

$$\begin{aligned} D_{p,\lambda}^n f^{(q)}(z) &= D(D_{p,\lambda}^{n-1} f^{(q)}(z)) \\ &= \frac{p!}{(p-q)!} z^{p-q} + \sum_{k-p+1}^{\infty} \frac{k!}{(k-q)!} \left(\frac{k+\lambda-q}{p+\lambda-q}\right)^n a_k z^{k-q} \\ &(n \in \mathbb{N},\ q \in \mathbb{N}_0 = \mathbb{N} \cup \{0\},\ \lambda \geqslant 0,\ p > q). \end{aligned} \qquad (1.11)$$

Special cases of this operator includes, the Aghalary derivative operator $D_{p,\lambda}^n\ f^{(0)}(z) = D_{p,\lambda}^n f(z)$ [1], the Cho and Kim derivative operator $D_{1,\lambda}^n f^{(0)}(z) = D_\lambda^n f(z)$ [2] and Salagean derivative operator $D_{1,0}^n f^{(0)}(z) = D^n$ [3]. Furthermore, we have

$$z(D_{p,\lambda}^n f^{(q)}(z))' = (p+\lambda-q) D_{p,\lambda}^{n+1} f^{(q)}(z) - \lambda D_{p,\lambda}^n f^{(q)}(z). \qquad (1.12)$$

Next by using the derivative operator $D_{p,\lambda}^n f^{(q)}(z)$, we introduce the following subclasses of $A(p)$

$$S^*[p,\lambda,q,n,\alpha] := \Big\{f : f \in A(p) \text{ and } D_{p,\lambda}^n f^{(q)}(z) \in S_p^*(\alpha)\ (0 \leqslant \alpha < p)\Big\}; \qquad (1.13)$$

$$C[p,\lambda,q,n,\alpha] := \Big\{f : f \in A(p) \text{ and } D_{p,\lambda}^n f^{(q)}(z) \in C_p(\alpha)\ (0 \leqslant \alpha < p)\Big\}; \qquad (1.14)$$

$$K_p[p,\lambda,q,n,\beta,\alpha] := \Big\{f : f \in A(p) \text{ and } D_{p,\lambda}^n f^{(q)}(z) \in K_p(\beta,\alpha)\ (0 \leqslant \alpha < p;\ 0 \leqslant \beta)\Big\}; \qquad (1.15)$$

And

$$K^*[p,\lambda,q,n,\beta,\alpha] := \Big\{f : f \in A(p) \text{ and } D^n_{p,\lambda} f^{(q)}(z) \in K^*_p(\beta,\alpha)\ (0 \leqslant \alpha < p;\ 0 \leqslant \beta)\Big\}.$$

To prove our main results, we need the following lemma which is popularly known as the Miller–Mocanu Lemma.

Lemma 1.1 (Miller and Mocanu [7]). *Let $\theta(\upsilon,\nu)$ be a complex-valued function such that*

$$\theta : \mathbb{D} \to \mathbb{C} \quad (\mathbb{D} \subset \mathbb{C} \times \mathbb{C}),$$

where $\mathbb{C}$ is complex plane, and let

$$\upsilon = \upsilon_1 + i\upsilon_2 \quad \textit{and} \quad \nu = \nu_1 + i\nu_2.$$

Suppose also that the function $\theta(\upsilon,\nu)$ satisfies each the following conditions:

(i) $\theta(\upsilon,\nu)$ *is continuous in* $\mathbb{D}$;

(ii) $(1,0) \in \mathbb{D}$ *and* $\mathrm{Re}(\theta(1,0)) > 0$;

(iii) $\mathrm{Re}(\theta(i\upsilon_2,\nu_1)) \leqslant 0$ *for all* $(i\upsilon_2,\nu_1) \in \mathbb{D}$ *such that*

$$\nu_1 \leqslant -\frac{1}{2}\big(1 + \upsilon_2^2\big).$$

Let

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \ldots \tag{1.16}$$

be analytic in U such that

$$(p(z), zp'(z)) \in \mathbb{D} \quad (z \in U).$$

If $\mathrm{Re}(\theta(p(z), zp'(z))) > 0$ $(z \in U)$, *then* $\mathrm{Re}(p(z)) > 0$ $(z \in U)$.

2. The Main Inclusion Relationships

In this section we will investigate several inclusion relationships for p-valent functions classes, which are associated the derivative operator $D^n_{p,\lambda} f^{(q)}(z)$. Our first theorem is the following

Theorem 2.1. *Let $f \in A(p)$. Suppose also that $(\lambda \geqslant 0,\ 0 \leqslant \alpha < p,\ p > q)$. Then*

$$S^*[p,\lambda,q,n+1,\alpha] \subset S^*[p,\lambda,q,n,\alpha] \quad (p+\lambda > q,\ 0 \leqslant \alpha < p). \tag{2.1}$$

$\lhd$ Let $f(z) \in S^*[p,\lambda,q,n+1,\alpha]$ and set

$$\frac{z\big(D^n_{p,\lambda} f^{(q)}(z)\big)'}{D^n_{p,\lambda} f^{(q)}(z)} = \alpha + (p-\alpha)p(z) \tag{2.2}$$

where $p(z)$ is given by (1.16) by applying the identity (1.12) we obtain

$$(p+\lambda-q)\left(\frac{D^{n+1}_{p,\lambda} f^{(q)}(z)}{D^n_{p,\lambda} f^{(q)}(z)}\right) = z\frac{\big(D^n_{p,\lambda} f^{(q)}(z)\big)'}{D^n_{p,\lambda} f^{(q)}(z)} + \lambda = (p-\alpha)p(z) + \alpha + \lambda.$$

By using logarithmic differentiation on both sides of the above equation, we have

$$\frac{z\big(D^{n+1}_{p,\lambda} f^{(q)}(z)\big)'}{D^{n+1}_{p,\lambda} f^{(q)}(z)} = \frac{z\big(D^{n}_{p,\lambda} f^{(q)}(z)\big)'}{D^{n}_{p,\lambda} f^{(q)}(z)} + \frac{(p-\alpha)zp'(z)}{(p-\alpha)p(z)+\alpha+\lambda}$$
$$= (p-\alpha)p(z)+\alpha+\frac{(p-\alpha)zp'(z)}{(p-\alpha)p(z)+\alpha+\lambda}.$$

We now choose $\upsilon = p(z) = \upsilon_1 + i\upsilon_2$ and $\nu = zp'(z) = \nu_1 + i\nu_2$, and define the function $\theta(\upsilon,\nu)$ by

$$\theta(\upsilon,\nu) = (p-\alpha)\upsilon + \frac{(p-\alpha)\nu_1}{(p-\alpha)\upsilon+\alpha+\lambda}. \tag{2.3}$$

Then, clearly, $\theta(\upsilon,\nu)$ is continuous in

$$\mathbb{D} = \left(\mathbb{C}\setminus\left\{\frac{\lambda+\alpha}{p-\alpha}\right\}\right)\times\mathbb{C} \quad \text{and} \quad (1,0)\in\mathbb{D} \text{ with } \operatorname{Re}(\theta(1,0))>0.$$

Moreover, for all $(i\upsilon_2,\nu_1)\in\mathbb{D}$ such that $\nu_1 \leqslant -\frac{1}{2}\big(1+\upsilon_2^2\big)$ we have

$$\operatorname{Re}(\theta(i\upsilon_2,\nu_1)) = \operatorname{Re}\left(\frac{(p-\alpha)\nu_1}{(p-\alpha)i\upsilon_2+\alpha+\lambda}\right),$$

$$\frac{(p-\alpha)(\alpha+\lambda)\nu_1}{(p-\alpha)\upsilon_2^2+(\alpha+\lambda)^2} \leqslant \frac{-(p-\alpha)(1+\upsilon_2^2)}{2\big(\big[(p-\alpha)\upsilon_2^2\big]^2+(\alpha+\lambda)^2\big)} < 0.$$

Which shows that $\theta(\upsilon,\nu)$ satisfies the conditions of Lemma 1.1.

This shows that if $\operatorname{Re}\theta(p(z), zp'(z)) > 0$ $(z\in U)$, then $\operatorname{Re} p(z) > 0$ $(z \in U)$, that is if $f^{(q)}(z) \in S^*[p,\lambda,q,n+1,\alpha]$ then $f^{(q)}(z)\in S^*[p,\lambda,q,n,\alpha]$. Then proof is of Theorem 2.1 is complete

Theorem 2.2. *Let $f \in A(p)$. Suppose also that $(\lambda \geqslant 0,\ 0\leqslant\alpha<p,\ p>q)$. Then $C[p,\lambda,q,n+1,\alpha]\subset C[p,\lambda,q,n+1,\alpha]$.*

$\lhd$ Let $f\in C[p,\lambda,q,n+1,\alpha]$. Then by (1.14), we have $(D^{n+1}_{p,\lambda} f^{(q)}(z))\in C_p(\alpha)$ furthermore, in view of the relationship (1.4) we find that

$$\frac{z}{p}\Big(D^{n+1}_{p,\lambda} f^{(q)}(z)\Big)' \in S_p^*(\alpha),$$

that is, that

$$D^{n+1}_{p,\lambda}\left(\frac{z}{p}\Big(f^{(q+1)}(z)\Big)\right)\in S_p^*(\alpha).$$

Thus by (1.13) and Theorem 2.1, we have

$$\frac{z}{p} f^{(q+1)}(z) \in S^*[p,\lambda,q,n+1,\alpha]\subset S^*[p,\lambda,q,n,\alpha],$$

which implies that

$$C[p,\lambda,q,n+1,\alpha]\subset C[p,\lambda,q,n,\alpha].$$

The proof of Theorem 2.2 thus complete. $\rhd$

Theorem 2.3. *Let $f\in A(p)$. Suppose also that $(\lambda\geqslant 0,\ 0\leqslant\alpha<p,\ p>q,\ \beta\geqslant 0)$. Then*

$$K[p,\lambda,q,n+1,\alpha]\subset K[p,\lambda,q,n,\alpha] \quad (p+\lambda>q,\ 0\leqslant\alpha<p,\ \beta\geqslant 0). \tag{2.4}$$

◁ Let $f(z) \in K[p,\lambda,q,n+1,\alpha]$. Then there exists a function $\psi(z) \in S_p^*(\alpha)$ such that

$$\mathrm{Re}\left(\frac{z(D_{p,\lambda}^{n+1}f^{(q)}(z))'}{\psi(z)}\right) > \beta \quad (z \in U).$$

We set $D_{p,\lambda}^{n+1}g^{(q)}(z) = \psi(z)$, so that we have

$$g(z) \in S^*[p,\lambda,q,n+1,\alpha] \quad \text{and} \quad \mathrm{Re}\left(\frac{z(D_{p,\lambda}^{n+1}f^{(q)}(z))'}{D_{p,\lambda}^{n+1}g^{(q)}(z)}\right) > \beta \quad (z \in U).$$

Now we put

$$\frac{z\big(D_{p,\lambda}^{n+1}f^{(q)}(z)\big)'}{D_{p,\lambda}^{n+1}g^{(q)}(z)} = \beta + (p-\beta)p(z), \tag{2.5}$$

where $p(z)$ is given, as before by (1.16) and using (1.12). From (2.5) we have

$$z(D_{p,\lambda}^{n+1}f^{(q)}(z))' = D_{p,\lambda}^{n+1}g^{(q)}(z)[\beta + (p-\beta)p(z)]. \tag{2.6}$$

$$\frac{z\big(D_{p,\lambda}^{n+1}f^{(q)}(z)\big)'}{D_{p,\lambda}^{n}g^{(q)}(z)} = \frac{D_{p,\lambda}^{n+1}\big(zf^{(q+1)}(z)\big)}{D_{p,\lambda}^{n+1}f^{(q)}(z)} = \frac{z\big[D_{p,\lambda}^{n}(zf^{(q+1)}(z))\big]' + \lambda\big(D_{p,\lambda}^{n}f^{(q+1)}(z)\big)}{z(D_{p,\lambda}^{n}g^{(q)}(z)) + \lambda D_{p,\lambda}^{n}g^{(q)}(z)}$$
$$= \frac{\frac{z\left[D_{p,\lambda}^{n}\left(zf^{(q+1)}(z)\right)\right]'}{D_{p,\lambda}^{n}g^{(q)}(z)} + \lambda\frac{D_{p,\lambda}^{n}\left(zf^{(q+1)}(z)\right)}{D_{p,\lambda}^{n}g^{(q)}(z)}}{\frac{z\left(D_{p,\lambda}^{n}g^{(q)}(z)\right)'}{D_{p,\lambda}^{n}g^{(q)}(z)} + \lambda}.$$

Since $g(z) \in S^*[p,\lambda,q,n+1,\alpha]$

$$\frac{z\big(D_{p,\lambda}^{n}g^{(q)}(z)\big)'}{D_{p,\lambda}^{n}g^{(q)}(z)} = \alpha + (p-\alpha)G(z),$$

where

$$G(z) = g_1(x,y) + ig_2(x,y) \quad \text{and} \quad \mathrm{Re}\,(G(z)) = g_1(x,y) > 0.$$

Then

$$\frac{z\big(D_{p,\lambda}^{n+1}f^{(q)}(z)\big)'}{D_{p,\lambda}^{n+1}g^{(q)}(z)} = \frac{\frac{\left(D_{p,\lambda}^{n}\left(zf^{(q+1)}(z)\right)\right)'}{D_{p,\lambda}^{n}g^{(q)}(z)} + \lambda[\beta + (p-\beta)p(z)]}{\alpha + (p-\alpha)G(z) + \lambda} \tag{2.7}$$

we get from (2.6) that

$$z\big(D_{p,\lambda}^{n}f^{(q)}(z)\big)' = D_{p,\lambda}^{n}g^{(q)}(z)[\beta + (p-\beta)p(z)]. \tag{2.8}$$

Upon differentiating both sides of (2.8) with respect to z we have

$$\frac{z\big[z\big(D_{p,\lambda}^{n}f^{(q)}(z)\big)'\big]'}{D_{p,\lambda}^{n}g^{(q)}(z)} = (p-\beta)zp'(z) + [\alpha + (p-\alpha)G(z)][\beta + (p-\beta)p(z)]. \tag{2.9}$$

By substituting (2.9) into (2.7) we obtain

$$\frac{z\big(D^{n+1}f^{(q)}(z)\big)'}{z\big(D_{p,\lambda}^{n+1}g(z)\big)} - \beta = (p-\beta)p(z) + \frac{(p-\beta)zp'(z)}{(p-\alpha)G(z) + \alpha + \lambda}$$

we now choose $\upsilon = p(z) = \upsilon_1 + i\upsilon_2$ and $\nu = zp'(z) = \nu_1 + i\nu_2$. If we defined the function $\theta(\upsilon, \nu)$ by

$$\theta(\upsilon,\nu) = (p-\beta)\upsilon + \frac{(p-\beta)\nu}{(p-\alpha)G(z)+\alpha+\lambda} \tag{2.10}$$

where

$$(\upsilon,\nu) \in \mathbb{D} = (\mathbb{C} \setminus \mathbb{D}^*) \times \mathbb{C}$$

and

$$\mathbb{D}^* := \left\{ z : \ z \in \mathbb{C} \quad \text{and} \quad \mathrm{Re}(G(z)) = g_1(x,y) \geqslant \frac{\lambda+\alpha}{p-\alpha} \right\}$$

it is easy to see that (υ, ν) is continuous in $\mathbb{D}$ and $(1,0) \in \mathbb{D}$ with $\mathrm{Re}(\theta(1,0)) > 0$. Moreover, for all $(i\upsilon_2, \nu_1) \in \mathbb{D}$ such that

$$\nu_1 \leqslant -\frac{1}{2}(1+\upsilon_2^2)$$

we have $\mathrm{Re}(\theta(1,0)) = \mathrm{Re}\left(\frac{(p-\beta)\nu_1}{(p-\alpha)G(z)+\alpha+\lambda}\right)$

$$\frac{(p-\beta)\nu_1[(p-\alpha)g_1(x,y)+\alpha+\lambda]}{[(p-\alpha)g_1(x,y)+\alpha+\lambda]^2 + [(p-\alpha)g_2(x,y)]^2}$$

$$\leqslant \frac{-(p-\beta)(1+\upsilon_2^2)[(p-\alpha)g_1(x,y)+\alpha+\lambda]}{2[(p-\alpha)g_1(x,y)+\alpha+\lambda]^2 + [(p-\alpha)g_2(x,y)]^2} < 0.$$

Which shows that $\theta(\upsilon, \nu)$ satisfies the conditions of Theorem 2.1. This completes the proof of Theorem 2.3.

Theorem 2.4. *Let $f \in A(p)$. Suppose also that $(\lambda \geqslant 0,\ 0 \leqslant \alpha < p,\ p > q,\ \beta \geqslant 0)$. Then*

$$K^*[p,\lambda,q,n+1,\alpha] \subset K^*[p,\lambda,q,n,\alpha] \quad (p+\lambda > q,\ 0 \leqslant \alpha < p,\ \beta \geqslant 0). \tag{2.11}$$

◁ We can prove Theorem 2.4 by using Theorem 2.3 in conjunction with the equation (1.7). Next we will study the integral operator given by [8]. ▷

3. Integral Operator

For $c > -p$ and $f(z) \in A(p)$ define the integral operator $J_{c,p}(f(z))$ as

$$J_{c,p}(f(z)) = \frac{c+p}{z^c} \int_0^z t^{c-1} f(t)\, dt. \tag{3.1}$$

The operator $J_{c,1}(f(z))$ $(c \in \mathbb{N})$ was introduced by Bernardi [4]. In particular, the operator $J_{1,1}(f(z))$ was introduced earlier by Libera [5] and Livingston [6].

Theorem 3.1. *Let $f \in A(p)$. Suppose also that $(\lambda \geqslant 0,\ 0 \leqslant \alpha < p,\ p > q$ and $c \geqslant -p)$. If $f(z) \in S^*[p,\lambda,q,n,\alpha]$ then $J_{c,p}(f(z))S^*[p,\lambda,q,n,\alpha]$.*

◁ Let $f(z) \in S^*[p,\lambda,q,n,\alpha]$. From (3.1) we have

$$z\Big(D^n_{p,\lambda} J_{c,p}\big(f^{(q)}(z)\big)\Big)' = (c+p) D^n_{p,\lambda} f^{(q)}(z) - c D^n_{p,\lambda} J_{c,p}(f^{(q)}(z)) \tag{3.2}$$

set

$$\frac{z\Big(D^n_{p,\lambda} J_{c,p}\big(f^{(q)}(z)\big)\Big)'}{D^n_{p,\lambda} J_{c,p}(f^{(q)}(z))} = \alpha + (p-\alpha)p(z). \tag{3.3}$$

Where $p(z)$ is given by (1.16) and using the identity (3.2) we have

$$\frac{D^n_{p,\lambda} f^{(q)}(z)}{D^n_{p,\lambda} J_{c,p}(f^{(q)}(z))} = \frac{1}{c+p}\{c+\alpha+(p-\alpha)p(z)\}. \tag{3.4}$$

Differentiating (3.4), we obtain

$$\frac{z\left(D^n_{p,\lambda} f^{(q)}(z)\right)'}{D^n_{p,\lambda} f^{(q)}(z)} - \alpha = (p-\alpha)p(z) + \frac{(p-\alpha)zp'(z)}{c+\alpha+(p-\alpha)p(z)}. \tag{3.5}$$

Now we define the function $\theta(\upsilon,\nu)$ by taking $\upsilon = p(z)$, $\nu = zp'(z)$ then (3.5) become as

$$\theta(\upsilon,\nu) = (p-\alpha)\upsilon + \frac{(p-\alpha)\nu}{c+\alpha+(p-\alpha)\upsilon}.$$

It is easy to see that the function $\theta(\upsilon,\nu)$ satisfies conditions (i) and (ii) of Lemma 1 in $\mathbb{D} = \left(\mathbb{C} \setminus \left\{\frac{c+\alpha}{p-\alpha}\right\}\right) \times \mathbb{C}$. Now we will prove third condition

$$\operatorname{Re}\{\theta(i\upsilon_2,\nu_1)\} = \operatorname{Re}\left\{\frac{(p-\alpha)\nu_1}{c+\alpha+(p-\alpha)i\upsilon_2}\right\}$$
$$= \frac{(p-\alpha)(c+\alpha)\nu_1}{(c+\alpha)^2+(p-\alpha)^2\upsilon_2^2} \leqslant \frac{-(p-\alpha)(c+\alpha)\left(1+\upsilon_2^2\right)}{2[(c+\alpha)^2+(p-\alpha)^2\upsilon_2^2]} < 0.$$

The function $\theta(\upsilon,\nu)$ satisfies conditions of Lemma 1.1. This shows that if $\operatorname{Re}\{\theta(p(z),zp'(z))\} > 0$ $(z \in U)$ then $\operatorname{Re} p(z) > 0$ $(z \in U)$, that is if $f(z) \in S^*[p,\lambda,q,n,\alpha]$. The prove is complete.

Theorem 3.2. *Let $f \in A(p)$. Suppose also that ($\lambda \geqslant 0$, $0 \leqslant \alpha < p$, $p > q$, $c \geqslant -p$). If $f(z) \in C[p,\lambda,q,n,\alpha]$ then $J_{c,p} \in C[p,\lambda,q,n,\alpha]$.*

$$\lhd\ f(z) \in C[p,\lambda,q,n,\alpha] \Rightarrow \frac{zf'(z)}{p} \in S^*[p,\lambda,q,n,\alpha] \Rightarrow J_{c,p}\frac{zf'(z)}{p} \in S^*[p,\lambda,q,n,\alpha]$$
$$\Leftrightarrow \frac{z}{p}J_{c,p}(f(z))' \in S^*[p,\lambda,q,n,\alpha] \Rightarrow J_{c,p}(f(z)) \in C[p,\lambda,q,n,\alpha].$$

This completes the proof of Theorem 3.2. $\rhd$

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Received April 15, 2011

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ВЛОЖЕНИЯ ДЛЯ НЕКОТОРЫХ ПОДКЛАССОВ p-ЛИСТНЫХ ФУНКЦИЙ, СВЯЗАННЫХ С ОБОБЩЕННЫМ ОПЕРАТОРОМ ДИФФЕРЕНЦИРОВАНИЯ

Эльджамал Э. А., Дарус М.

Вводятся новые классы аналитических p-листных функций, определяемые обобщенным оператором дифференцирования, и изучаются различные вложения этих классов. Рассматриваются некоторые интересные приложения, включая классы интегральных операторов.

Ключевые слова: p-листная функция, оператор дифференцирования, интегральный оператор, свойство вложения.