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BLOCK GRAPH OF A GRAPH

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Abstract. The *block graph* of a graph G, written $B(G)$, is the graph whose vertices are the blocks of G and in which two vertices are adjacent whenever the corresponding blocks have a cut-vertex in common. We study the properties of $B(G)$ and present the characterization of graphs whose $B(G)$ are planar, outerplanar, maximal outerplanar, minimally non-outerplanar, Eulerian, and Hamiltonian. A necessary and sufficient condition for $B(G)$ to have crossing number one is also presented.

Key words: crossing number, inner vertex number, dutch windmill graph, complete graph.

Mathematical Subject Classification (2010): 05C05, 05C45.

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1. Introduction

Notations and definitions not introduced here can be found in [1]. There are many graph operators (or graph valued functions) with which one can construct a new graph from a given graph, such as the line graph, the total graph, and their generalizations. One such generalization is the block graph concept whose properties and characterizations were considered in [2]. It is the object of this paper to study some of the structural properties of the block graph such as the planarity, outer planarity, etc.

We need some concepts and notations on graphs. A graph $G = (V, E)$ is a pair, consisting of some set V, the so-called *vertex set*, and some subset E of the set of all 2-element subsets of V, the *edge set*. We write $x = (p, q)$ and say that p and q are *adjacent vertices* (sometimes denoted p *adj* q).

A graph G is *connected* if between any two distinct vertices there is a path. A *maximal connected subgraph* of G is called a *component* of G. A *cut-vertex* of a graph is one whose removal increases the number of components. A *non-separable* graph is connected, nontrivial, and has no cut-vertices. A *block* of a graph is a maximal non-separable subgraph. If two distinct blocks B_1 and B_2 are incident with a common cut-vertex, then they are called *adjacent blocks*.

A graph G is *planar* if it has a drawing without crossings. For a planar graph G, the *inner vertex number* $i(G)$ is the minimum number of vertices not belonging to the boundary of the exterior region in any embedding of G in the plane.

If a planar graph G is embeddable in the plane so that all the vertices are on the boundary of the exterior region, then G is said to be *outerplanar*. An outerplanar graph G is *maximal outerplanar* if no edge can be added without losing outerplanarity. A *grap* G is said to be *minimally non-outerplanar* if $i(G) = 1$ [3]. The least number of edge crossings of a graph G, among all planar embeddings of G, is called the *crossing number* of G and is denoted by $cr(G)$.

A *star graph* $K_{1,n}$ $(n \geqslant 3)$, is the complete bipartite graph. The *dutch windmill graph* $D_3^{(m)}$, also called a *friendship graph*, is the graph obtained by taking m copies of the cycle graph C_3 with a vertex in common and therefore corresponds to the usual windmill graph $D_3^{(m)}$. It is therefore natural to extend the definition to $D_n^{(m)}$, consisting of m copies of C_n.

Definition 1.1. The *line graph* of a graph G, written $L(G)$, is the graph whose vertices are the edges of G, with two vertices of $L(G)$ adjacent whenever the corresponding edges of G have a common vertex.

Definition 1.2. The *block graph* of a graph G, written $B(G)$, is the graph whose vertices are the blocks of G and in which two vertices are adjacent whenever the corresponding blocks have a cut-vertex in common.

Note that $B(G)$ is defined only for graphs which have at least one cut-vertex or (at least two blocks). In Fig. 1, a graph G and its $B(G)$ are shown.

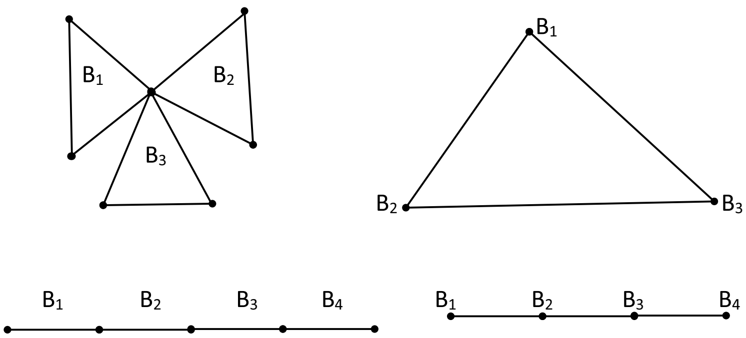

Fig. 1.

2. Properties of Block Graphs

In this section we present some of the basic properties of $B(G)$.

Property 2.1. If G is a tree of order n $(n \geqslant 3)$, then $L(G) \cong B(G)$.

Property 2.2. There is no non-trivial graph G which is isomorphic to its $B(G)$.

Property 2.3. The block graph $B(G)$ of a graph G is a block if G contains exactly one cut-vertex.

Property 2.4. If the number of cut-vertices of a path P_n $(n \geqslant 3)$ is α, then number of cut-vertices of the corresponding $B(P_n)$ is $\alpha - 1$. Clearly, the number of cut-vertices of $B(K_{1,n})$ is zero.

Property 2.5. If G is a path P_n $(n \geqslant 2)$, then the size of $B(P_n)$ equals $\frac{1}{2}\sum_{i=1}^{n} d_i^2 - n + 1$, where d_i is the degree of the vertices of P_n.

Property 2.6. If G is a star graph $K_{1,n}$ $(n \geqslant 3)$, then the size of $B(K_{1,n}) = \frac{n(n-1)}{2}$.

3. Characterization of $B(G)$

We now characterize the graphs whose $B(G)$ are planar.

Theorem 3.1. *The block graph $B(G)$ of a graph G is planar if and only if G is either a star graph $K_{1,n}$ $(2 \leqslant n \leqslant 4)$ or a dutch windmill graph $D_n^{(m)}$ $(2 \leqslant m \leqslant 4)$.*

$\lhd$ Suppose that $B(G)$ is planar. Assume that $G = K_{1,n}$ $(n \geqslant 5)$. If $G = K_{1,5}$, then $B(G) = K_5$, which is non-planar, a contradiction. Assume now that $G = D_n^{(m)}$ $(m \geqslant 5)$. If $G = D_n^{(5)}$, then $B(G) = K_5$, again a contradiction.

Conversely, suppose that G is either a star graph $K_{1,n}$ $(2 \leqslant n \leqslant 4)$ or a dutch windmill graph $D_n^{(m)}$ $(2 \leqslant m \leqslant 4)$. We consider the following cases.

Case 1: If $G = K_{1,2}$, then $B(G) = K_2$, which is planar.

Case 2: If $G = K_{1,3}$, then $B(G) = K_3$, which is planar.

Case 3: If $G = K_{1,4}$, then $B(G) = K_4$, which is planar.

Case 4: If $G = D_n^{(2)}$, then $B(G) = K_2$, which is planar.

Case 5: If $G = D_n^{(3)}$, then $B(G) = K_3$, which is planar.

Case 6: If $G = D_n^{(4)}$, then $B(G) = K_4$, which is planar.

Therefore, by all the cases above, $B(G)$ is planar. This completes the proof. $\rhd$

We now establish a characterization of graphs whose $B(G)$ are outerplanar; maximal outerplanar; and minimally non-outerplanar.

Theorem 3.2. *The block graph $B(G)$ of a graph G is outerplanar if and only if G is either $K_{1,3}$ or $D_n^{(3)}$.*

$\lhd$ Suppose that $B(G)$ is outerplanar. Assume that G is either $K_{1,n}$ $(n \geqslant 4)$ or $D_n^{(m)}$ $(m \geqslant 4)$. If $G = K_{1,4}$, then $B(G) = K_4$. Clearly the inner vertex number of $B(G)$ is one, i. e., $i(B(G)) = 1$, a contradiction. Assume now that $G = D_n^{(m)}$ $(m \geqslant 4)$. If $G = D_n^{(4)}$, then $B(G) = K_4$, again a contradiction.

Conversely, suppose that G is either $K_{1,3}$ or $D_n^{(3)}$. If $G = K_{1,3}$, then $B(G) = K_3$. Clearly the inner vertex number of $B(G)$ is zero, i. e., $i(B(G)) = 0$. If $G = D_n^{(3)}$, then $B(G) = K_3$, and thus $i(B(G)) = 0$. Therefore, $B(G)$ is outerplanar. This completes the proof. $\rhd$

Theorem 3.3. *The block graph $B(G)$ of a graph G is maximal outerplanar if and only if G is either $K_{1,3}$ or a path P_3.*

$\lhd$ Suppose that $B(G)$ is maximal outerplanar. Assume that G is $K_{1,n}$ $(n \geqslant 4)$. If $G = K_{1,4}$, then $B(G) = K_4$, which is non-outerplanar, a contradiction. Assume now that G is a path P_n of order n $(n \geqslant 4)$. By definition, $B(G)$ is a path of order $n - 1$. Clearly, $i(B(G)) = 0$, and the addition of an edge does not change the inner vertex number of $B(G)$. Clearly, $B(G)$ is not maximal outerplanar, again a contradiction.

Conversely, suppose that G is either $K_{1,3}$ or a path P_3. If $G = K_{1,3}$, then $B(G) = K_3$, which is maximal outerplanar. If $G = P_3$, then $B(G) = P_2$, which is also maximal outerplanar. This completes the proof. $\rhd$

Theorem 3.4. *The block graph $B(G)$ is minimally non-outerplanar if and only if G is either $K_{1,4}$ or $D_n^{(4)}$.*

$\lhd$ Suppose $B(G)$ is minimally non-outerplanar. Assume that $G = K_{1,5}$. By definition, $B(G) = K_5$, which is non-planar, a contradiction. On the other hand, if $G = D_n^{(5)}$, then $B(G) = K_5$, again a contradiction.

Conversely, suppose that G is either $K_{1,4}$ or $D_n^{(4)}$. By definition, $B(G) = K_4$. Clearly, $i(B(G)) = 1$. Hence $B(G)$ is minimally non-outerplanar. This completes the proof. ▷

Theorem 3.5. *The block graph $B(G)$ of a graph G has crossing number one if and only if G is either $K_{1,5}$ or $D_n^{(5)}$.*

◁ Suppose G has crossing number one. Assume that $G = K_{1,n}$ $(n \geqslant 6)$. If $G = K_{1,6}$, then $B(G) = K_6$. Clearly, $cr(B(G)) > 1$, a contradiction. On the other hand, if $G = D_n^{(6)}$, then $B(G) = K_6$, a contradiction.

Conversely, suppose that G is either $K_{1,5}$ or $D_n^{(5)}$. By definition, $B(G) = K_5$. Since the crossing number of K_5 is exactly one, $cr(B(G)) = 1$. This completes the proof. ▷

Definition 3.1. An *Eulerian cycle* in an undirected graph is a cycle that uses each edge exactly once. If such a cycle exists, then the graph is called *Eulerian*.

Theorem 3.6 (Harary [1]). *A connected graph G is said to be Eulerian if and only if the degree of each vertex of G is even.*

Theorem 3.7. *The block graph $B(G)$ of a graph G is Eulerian if and only if G is either $K_{1,2k+1}$ or $D_n^{(2k+1)}$ $(k \geqslant 1)$.*

◁ Suppose $B(G)$ is Eulerian. Assume that $G = K_{1,2k}$ $(k \geqslant 1)$. By definition, $B(G) = K_{2k}$ in which degree of each vertex is $2k-1$, which is odd. Since the degree of each vertex of $B(G)$ is odd, Theorem 3.6 implies that $B(G)$ is non-Eulerian, a contradiction. On the other hand, if $G = D_n^{(2k)}$, then $B(G) = K_{2k}$, again a contradiction.

Conversely, suppose that G is either $K_{1,2k+1}$ or $D_n^{(2k+1)}$ $(k \geqslant 1)$. By definition, $B(G) = K_{2k+1}$, in which the degree of each vertex of $B(G)$ is $2k$, which is even for every $k \geqslant 1$. Since the degree of each vertex of $B(G)$ is even, Theorem 3.6 implies that $B(G)$ is Eulerian. This completes the proof. ▷

Definition 3.2. A *Hamiltonian path* is a path that visits each vertex of the graph exactly one. A graph is *Hamiltonian* if for every pair of vertices there is a Hamiltonian path between the two vertices.

Theorem 3.8. *The block graph $B(G)$ of $K_{1,n}$ $(n \geqslant 3)$ or $D_n^{(m)}$ $(m \geqslant 3)$ is Hamiltonian.*

◁ Suppose that G is either $K_{1,n}$ $(n \geqslant 3)$ or $D_n^{(m)}$ $(m \geqslant 3)$. By definition, $B(G)$ is a complete graph of order n or m. Since every complete graph is Hamiltonian, $B(G)$ is Hamiltonian. This completes the proof. ▷

4. Open problems

4.1. One can naturally extend these concepts to the directed graph version. What can one say about the properties of the directed version?

4.2. If the number of cut-vertices of the graph G is β, then what is the number of cut-vertices of the corresponding $B(G)$?

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ГРАФ БЛОКОВ

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Аннотация. Граф блоков $B(G)$ графа G — граф, вершинами которого являются блоки графа G и в котором две вершины смежны тогда и только тогда, когда соответствующие им блоки имеют общую точку сочленения. Изучаются различные свойства графа блоков $B(G)$, в частности, даны характеристики графов, у которых графы блоков $B(G)$ являются плоскими (планарными), внешнепланарными, максимальными внешнепланарными, минимальными невнешнепланарными, эйлеровыми и гамильтоновыми. Также представлено необходимое и достаточное условие, чтобы число пересечения графа блоков $B(G)$ равнялось единице.

Ключевые слова: число пересечения, число внутренних вершин, граф «голландская мельница», полный граф.

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