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A KRENGEL TYPE THEOREM FOR COMPACT OPERATORS BETWEEN LOCALLY SOLID VECTOR LATTICES

O. Zabeti[1]

[1] Department of Mathematics, Faculty of Mathematics, Statistics and Computer Science, University of Sistan and Baluchestan, Zahedan, P.O. Box 98135-674, Iran

E-mail: o.zabeti@gmail.com

Abstract. Suppose X and Y are locally solid vector lattices. A linear operator $T : X \to Y$ is said to be nb-compact provided that there exists a zero neighborhood $U \subseteq X$, such that $\overline{T(U)}$ is compact in Y; T is bb-compact if for each bounded set $B \subseteq X$, $\overline{T(B)}$ is compact. These notions are far from being equivalent, in general. In this paper, we introduce the notion of a locally solid AM-space as an extension for AM-spaces in Banach lattices. With the aid of this concept, we establish a variant of the known Krengel's theorem for different types of compact operators between locally solid vector lattices. This extends [1, Theorem 5.7] (established for compact operators between Banach lattices) to different classes of compact operators between locally solid vector lattices.

Keywords: compact operator, the Krengel theorem, locally solid AM-space.

AMS Subject Classification: 46B42, 47B65.

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1. Motivation and Introduction

Let us start with some motivation. Let E be a Banach lattice. E is called an AM-space provided that for each $x, y \in E_+$, we have $\|x \vee y\| = \|x\| \vee \|y\|$. The remarkable Kakutani theorem states that every AM-space is a closed sublattice of some $C(K)$-space, in which K is the compact Hausdorff topological space. Now, suppose E is a Banach lattice and F is an AM-space. The Krengel theorem states that every compact operator $T : E \to F$ has a modulus which is defined by the Riesz–Kantorovich formulae; that is $|T|(x) = \sup\{|Ty| : |y| \leqslant x\}$ for each $x \in E_+$; furthermore, $|T|$ is also compact. So, we conclude that AM-spaces have many interesting properties among the category of all Banach lattices. Therefore, it is fascinating and significant to consider the AM-spaces and numerous applications in the operator theory for locally solid vector lattices and operators between them. In this paper, we consider locally solid vector lattices whose family of pseudonorms which generate the topology of X, preserve the finite suprema; we call them: locally solid AM-spaces. This definition extends AM-spaces to the category of all locally solid vector lattices. Note that a variant of this notion has been defined by the author in [2]; however, that definition has a mild gap so that we consider the new definition using the generating pseudonorms. Moreover, observe that there are several different ways to define bounded and compact operators between locally solid vector lattices.

In this paper, we are going to generalize the known Krengel theorem [1, Theorem 5.7] for different types of compact operators between locally solid vector lattices.

Note that a subset A of a vector lattice X is called solid provided that $x \in X$, $y \in A$ and $|x| \leqslant |y|$ implies that $x \in A$. Moreover, a pseudonorm ρ on a vector lattice X is a real-valued function on X that satisfies the following properties.

(i) $\rho(x) \geqslant 0$ *for each* $x \in X$;

(ii) $\rho(x + y) \leqslant \rho(x) + \rho(y)$ *for each* $x, y \in X$;

(iii) $\rho(\lambda x) \to 0$ *as* $\lambda \to 0$, *for each* $x \in X$;

(iv) $\rho(x) \leqslant \rho(y)$ *whenever* $|x| \leqslant |y|$ *for each* $x, y \in X$.

For undefined terminology and related notions (see [1, 3]). All locally solid vector lattices in this note are assumed to be Hausdorff.

2. Main Result

We introduce the notion of the locally solid AM-space; a variant of this notion has been defined as the AM-property in [2]; first, let us recall the AM-property.

Suppose X is a locally solid vector lattice. We say that X has the AM-property provided that for every bounded set $B \subseteq X$, $B^{\vee}$ is also bounded with the same scalars; namely, given a zero neighborhood V and any positive scalar α with $B \subseteq \alpha V$, we have $B^{\vee} \subseteq \alpha V$. Note that by $B^{\vee}$, we mean the set of all finite suprema of elements of B.

That definition has a gap as follows. Let X be $\mathbb{R}^2$ with the sup norm. Take $B = \{(1, 1), (\sqrt{2}, 0)\}$ and $V = \{(x, y) : x^2 + y^2 \leqslant 1\}$. Note that V is a neighbourhood of 0. Then B is contained in $\sqrt{2}V$, but $B^{\vee}$ is not.

On the other hand, by [3, Theorem 2.28] due to Fremlin, the linear topology of every locally solid vector lattice has been generated by a family of the Riesz pseudonorms. So, we can define a locally solid AM-space as follows. A version of the following definition was originally defined at first in [4].

DEFINITION 1. A locally solid AM-space is a locally solid space X together with a family $(\rho_i)_{i \in I}$ of the Riesz pseudonorms which generate the topology of X and, in addition, satisfy the following property:

$$\rho_i(x \vee y) = \rho_i(x) \vee \rho_i(y)$$

for each $i \in I$ and for each $x, y \in X_+$.

This definition modifies the gap in definition of the AM-property considered in [2]. Now, we restate the following useful fact which is proved initially in [5, Lemma 3] while we employed the AM-property instead of the locally solid AM-space.

Lemma 1. *Suppose X is a locally solid AM-space with a family $(\rho_i)_{i \in I}$ of the Riesz pseudonorms, which generate the topology of X and U is an arbitrary solid zero neighborhood in X. Then, for each $m \in \mathbb{N}$, $U \vee \ldots \vee U = U$, in which U is appeared m-times.*

$\lhd$ Without loss of generality, we may assume that

$$U = \{x \in X,\ \rho_{i_k}(x) < \varepsilon_k :\ 1 \leqslant k \leqslant n\}$$

in which $\{i_1, \ldots, i_n\} \subseteq I$ and $\{\varepsilon_1, \ldots, \varepsilon_n\} \subseteq \mathbb{R}_+$. It is obvious that $U \subseteq U \vee \ldots \vee U$. For the other direction, assume that $x_1, \ldots, x_m \in U_+$. For each $1 \leqslant k \leqslant n$, we have $\rho_k(x_1, \vee \ldots \vee x_m) = \rho_k(x_1) \vee \ldots \vee \rho_k(x_m) < \varepsilon_k$. This shows that $U \vee \ldots \vee U \subseteq U$, as claimed. $\rhd$

Moreover, we have the following useful inequality in the Archimedean vector lattices.

Lemma 2. *Suppose E is a vector lattice. Then for $x_1, \dots, x_n$ and $y_1, \dots, y_n$ in E, the following inequality holds*

$$x_1 \vee \dots \vee x_n - y_1 \vee \dots \vee y_n \leqslant (x_1 - y_1) \vee \dots \vee (x_n - y_n).$$

$\lhd$ We proceed the proof by induction. For $n = 2$, we have

$$\begin{aligned} x_1 \vee x_2 - y_1 \vee y_2 &= (x_1 - (y_1 \vee y_2)) \vee (x_2 - (y_1 \vee y_2)) \\ &= (x_1 + ((-y_1) \wedge (-y_2))) \vee (x_2 + ((-y_1) \wedge (-y_2))) \\ &= ((x_1 - y_1) \wedge (x_1 - y_2)) \vee ((x_2 - y_1) \wedge (x_2 - y_2)) \leqslant (x_1 - y_1) \vee (x_2 - y_2). \end{aligned}$$

Now, suppose for $n = k$, the statement is valid. We need prove it for $n = k + 1$. By using validness of the result for $n = 2$ and $n = k$, we have

$$\begin{aligned} & x_1 \vee \dots \vee x_k \vee x_{k+1} - y_1 \vee \dots \vee y_k \vee y_{k+1} \\ & \leqslant ((x_1 \vee \dots \vee x_k) - (y_1 \vee \dots \vee y_k)) \vee (x_{k+1} - y_{k+1}) \\ & \leqslant (x_1 - y_1) \vee \dots \vee (x_k - y_k) \vee (x_{k+1} - y_{k+1}). \rhd \end{aligned}$$

Recall that a subset B of a topological vector space X is said to be totally bounded, if for each arbitrary zero neighborhood $V \subseteq X$ there is a finite set F such that $B \subseteq F + V$; for more information, see [1]. Another proof of the following result with a different technique has been obtained in [6, Corollary 4.3.5].

Lemma 3. *Suppose X is a topologically complete locally solid AM-space. If $B \subseteq X$ is totally bounded, then so is $B^{\vee}$. In particular, $\sup B$ exists in X and $\sup B \in \overline{B^{\vee}}$.*

$\lhd$ Choose arbitrary solid zero neighborhood $U \subseteq X$. By the assumption, there exists a finite set $F \subseteq X$ such that $B \subseteq F + U$. Assume that $F = \{z_1, \dots, z_m\}$. We claim that $B^{\vee} \subseteq F^{\vee} + U$; note that $F^{\vee}$, the set of all finite suprema of elements of F, is clearly finite. Given any $x_1, \dots, x_n \in B$. There are some $z_1, \dots, z_n$ (possibly with the repetition), such that $x_i - z_i \in U$ for all $i = 1, \dots, n$. Therefore, by using Lemma 2 and Lemma 1, we have

$$x_1 \vee \dots \vee x_n - z_1 \vee \dots \vee z_n \leqslant (-z_1 + x_1) \vee \dots \vee (-z_n + x_n) \in U \vee \dots \vee U = U.$$

Since U is solid, similarly, we have

$$z_1 \vee \dots \vee z_n - x_1 \vee \dots \vee x_n \leqslant (z_1 - x_1) \vee \dots \vee (z_n - x_n) \in U \vee \dots \vee U = U.$$

This means that $(x_1 \vee \dots \vee x_n) - (z_1 \vee \dots \vee z_n) \in U$ so that $x_1 \vee \dots \vee x_n \in F^{\vee} + U$.

Now, assume that D is the set of all finite subsets of B directed by the inclusion $\subseteq$. For each $\alpha \in D$, put $g_\alpha = \sup \alpha$. Observe that $\{g_\alpha\} \subseteq B^{\vee}$ satisfies $g_\alpha \uparrow$. By compactness of $\overline{B^{\vee}}$, there exists a subnet of (g_α) that converges to some $g \in \overline{B^{\vee}}$. Therefore, $\sup B = \sup B^{\vee} = \sup\{g_\alpha\} = g$. $\rhd$

Now, we are able to consider a version of the Krengle's theorem [1, Theorem 5.7] for each class of compact operators between locally solid vector lattices. First, we recall some preliminaries which are needed in the sequel.

Suppose X and Y are locally solid vector lattices and $T : X \to Y$ is a linear operator. T is called *nb*-bounded, if there is a zero neighborhood $U \subseteq X$, such that $T(U)$ is also bounded in Y; T is said to be *bb*-bounded, if it maps bounded sets into bounded sets.

Moreover, a linear operator $T : X \to Y$ is said to be *nb*-compact provided that there is a zero neighborhood $U \subseteq X$, such that $\overline{T(U)}$ is compact in Y; T is *bb*-compact if for every bounded set $B \subseteq X$, $\overline{T(B)}$ is compact in Y. It is obvious that every *nb*-compact operator is *nb*-bounded and every *bb*-compact operator in *bb*-bounded. These classes of operators enjoy some topological and lattice structures; for a detailed exposition as well as related notions about bounded and compact operators see [2, 7, 8].

Krengel has proved that when the range of a compact operator T between the Banach lattices is an AM-space, then the modulus of T exists and is also compact (see [1, Theorem 5.7]). In the following, we prove this remarkable result for *nb*-compact operators as well as for *bb*-compact operators, when the range space is a locally solid AM-space.

Theorem 1. *Suppose X is a locally solid vector lattice, Y is a topologically complete locally solid AM-space and $T : X \to Y$ is a bb-compact operator. Then the modulus of T exists and is also bb-compact.*

$\lhd$ Fix a bounded set $B \subseteq X$ such that $T(B)$ is totally bounded in Y; by replacing B with $Sol(B)$, if necessary, we may assume that B is solid. Observe that for each $x \in B_+$, $T[-x, x]$ is totally bounded in Y so that by Lemma 3, the supremum $|T|(x) = \sup\{|Ty| : |y| \leqslant x\} = \sup T[-x, x]$ exists in Y. Thus, by [1, Theorem 1.14], the modulus of T exists. According to Lemma 3, $\overline{T(B)^{\vee}}$ is also compact and $|T|(x) \in \overline{T(B)^{\vee}}$. Therefore, $|T|(B_+) \subseteq \overline{T(B)^{\vee}}$. Since $B \subseteq B_+ - B_+$, we have the desired result. $\rhd$

Corollary 1. *Suppose X is a locally solid vector lattice, Y is a topologically complete locally solid AM-space and $T : X \to Y$ is an nb-compact operator. Then the modulus of T exists and is also nb-compact.*

$\lhd$ Observe that every *nb*-compact operator is *bb*-compact. Therefore, by Theorem 1, the modulus of T exists. We need to show that it is also *nb*-compact. There exists a zero neighborhood $U \subseteq X$, such that $T(U)$ is totally bounded in Y. Note that according to Lemma 3, $\overline{T(U)^{\vee}}$ is also compact and $|T|(x) \in \overline{T(U)^{\vee}}$. Therefore, $|T|(U_+) \subseteq \overline{T(U)^{\vee}}$. Since $U \subseteq U_+ - U_+$, the proof would be complete. $\rhd$

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Omid Zabeti
Department of Mathematics, Faculty of Mathematics, Statistics, and Computer Science, University of Sistan and Baluchestan, Zahedan, P.O. Box 98135-674, Iran,
Associate Professor
E-mail: `o.zabeti@gmail.com`
https://orcid.org/0000-0001-6875-2418

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ТИП ТЕОРЕМЫ КРЕНГЕЛЯ ДЛЯ КОМПАКТНЫХ ОПЕРАТОРОВ МЕЖДУ ЛОКАЛЬНО ПЛОТНЫМИ ВЕКТОРНЫМИ РЕШЕТКАМИ

Забети О.[1]

[1] Университет Систана и Белуджистана,
Иран, 98155-987, Захедан
E-mail: o.zabeti@gmail.com

Аннотация. Предположим, что X и Y — локально плотные векторные решетки. Линейный оператор $T : X \to Y$ называется nb-компактным, если существует нулевая окрестность $U \subseteq X$ такая, что оператор $\overline{T(U)}$ компактен в Y. Оператор T bb-компактен, если для любого ограниченного множества $B \subseteq X$ $\overline{T(B)}$ компактно. Эти понятия далеко не равнозначны, вообще говоря. В этой статье мы вводим понятие локально плотного AM-пространства как расширения для AM-пространств в банаховых решетках. С помощью этого понятия устанавливается вариант известной теоремы Кренгеля для различных типов компактных операторов между локально плотными векторными решетками. Эта теорема распространяется [1, Теорема 5.7] (установленную для компактных операторов между банаховыми решетками) на различные классы компактных операторов между локально телесными векторными решетками.

Ключевые слова: компактный оператор, теорема Кренгеля, локально плотное AM-пространство.

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