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INVERSE COEFFICIENT PROBLEM FOR THE 2D WAVE EQUATION WITH INITIAL AND NONLOCAL BOUNDARY CONDITIONS

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Dedicated to prof. G. G. Magaril-Ilyaev
on the occasion of his 80th birthday

Abstract. In this paper, we consider direct and inverse problems for the two-dimensional wave equation. The direct problem is an initial boundary value problem for this equation with nonlocal boundary conditions. In the inverse problem, it is required to find the time-variable coefficient at the lower term of the equation. The classical solution of the direct problem is presented in the form of a biorthogonal series in eigenvalues and associated functions, and the uniqueness and stability of this solution are proven. For solution to the inverse problem, theorems of existence in local, uniqueness in global, and an estimate of conditional stability are obtained. The problems of determining the right-hand sides and variable coefficients at the lower terms from initial boundary value problems for second-order linear partial differential equations with local boundary conditions have been studied by many authors. Since the nonlinearity is convolutional, the unique solvability theorems in them are proven in a global sense. In the works, the method of separation of variables is used to find the classical solution of the direct problem in the form of a biorthogonal series in terms of eigenfunctions and associated functions. The nonlocal integral condition is used as the overdetermination condition with respect to the solution of the direct problem. The direct problem reduces to equivalent integral equations of the Fourier method. To establish integral inequalities, the generalized Gronwall-Bellman inequality is used. We obtain an a priori estimate of the solution in terms of an unknown coefficient, that are useful for studying the inverse problem.

Keywords: wave equation, nonlocal boundary conditions, inverse problem, Banach's theorem.

AMS Subject Classification: 35P05, 47F05, 35A01, 35A24, 35J05 , 35P30.

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1. Formulation of the Problem

Inverse problems for partial differential equations are understood as the problems of finding unknown coefficients, as well as initial and non-local boundary conditions and solutions of differential equations given the solution of a direct problem. Inverse problems are a dynamically developing area of modern mathematics. A lot of papers are devoted to inverse problems for second-order hyperbolic equations (see, for example, monographs [1–4] and references therein).

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The problems of determining the right-hand sides and variable coefficients at the lower terms from initial boundary value problems for second-order linear partial differential equations with local boundary conditions have been studied by many authors. Among them, as the closest to the present work on the research method, we note the works [5–7], in which existence and uniqueness theorems are formulated in the global sense for problems of determining the coefficients of an equation (nonlinear problem). In this vein, we aslo note papers [8–13] realted to the inverse problems of recovering kernels in hyperbolic integro-differential equations. Since the nonlinearity is convolutional, the unique solvability theorems in them are proven in a global sense.

Problems with non-local boundary conditions for partial differential equations have been studied by many authors, starting with the already classic work [14]. A specific feature of non-local problems is the non-self-adjointness of the spatial differential operator and, as a consequence, the incompleteness of the system of eigenfunctions, which has to be supplemented by adjoint functions. Fundamental results on the basis property of a system of eigenfunctions and associated functions were obtained in [15, 16].

In the works, the method of separation of variables is used to find the classical solution of the direct problem in the form of a biorthogonal series in terms of eigenfunctions and associated functions. The nonlocal integral condition is used as the overdetermination condition with respect to the solution of the direct problem. The direct problem reduces to equivalent integral equations by the Fourier method. To establish integral inequalities, the theorems of the generalized Gronwall–Bellman inequality is used. We obtain an a priori estimate of the solution in terms of an unknown coefficient, which we will need to study the inverse problem. The inverse problem is reduced to the Volterra integral equation of the second kind. On the basis of the unique solvability of this equation in the class of continuous functions, theorems on the unique solvability of direct and inverse problems are proved. A stability estimate is also obtained.

Let $\Omega = D \times (0,T)$, where $D = \{(x,y) : 0 < x, y < 1\}$. We consider the initial boundary value problem for wave equation

$$u_{tt}(x,y,t) - \Delta u + q(t)u(x,y,t) = f(x,y,t), \tag{1}$$

with initial conditions

$$u(x,y,t)\big|_{t=0} = \varphi_1(x,y), \quad (x,y) \in \overline{D}, \tag{2}$$

$$u_t(x,y,t)\big|_{t=0} = \varphi_2(x,y), \quad (x,y) \in \overline{D}, \tag{3}$$

and non-local boundary conditions

$$u_x(0,y,t) = u_x(1,y,t), \quad u(0,y,t) = 0, \quad (y,t) \in [0,1] \times [0,T], \tag{4}$$

$$u(x,0,t) = u(x,1,t), \quad u_y(x,1,t) = 0, \quad (x,t) \in [0,1] \times [0,T], \tag{5}$$

where $\overline{D} = \{(x,y) : 0 \leqslant x, y \leqslant 1\}$, $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$; $\varphi_1(x,y)$, $\varphi_2(x,y)$ and $f(x,y,t)$ are given functions.

In the direct problem, it is required to define a function $u(x,y,t) \in C^2(\overline{\Omega})$ satisfying the equalities (1)–(5), for given sufficiently smooth functions $q(t)$, $f(x,y,t)$, $\varphi_i(x,y)$, $i = 1,2$, where $\overline{\Omega} = \{(x,y,t) : 0 \leqslant x,\ y \leqslant 1,\ 0 \leqslant t \leqslant T\}$.

The inverse problem is to find function $q(t) \in C[0,T]$, if with respect to the solution of the direct problem (1)–(5) the overdetermination condition is known

$$\int_0^1 \int_0^1 w(x,y)u(x,y,t)\,dxdy = h(t), \tag{6}$$

where $h(t)$, $w(x, y)$ are given functions. We use the following lemma to interpret the direct problem.

Lemma 1 [17]**.** *Let a non-negative function continuous on $[c, d]$ and $u(t)$ satisfy the inequality*

$$u(t) \leqslant a(t) + b(t) \int_c^t k(t, s)u(s)\, ds,$$

where $a(t) \geqslant 0$, $b(t) \geqslant 0$, $k(t, s) \geqslant 0$ — continuous functions on $c \leqslant s \leqslant t \leqslant d$. Then

$$u(t) \leqslant A(t) \exp\left\{ B(t) \int_c^t K(t, s)\, ds \right\},$$

here

$$A(t) = \sup_{c \leqslant s \leqslant t} a(s), \quad B(t) = \sup_{c \leqslant s \leqslant t} b(s), \quad K(t, s) = \sup_{s \leqslant \sigma \leqslant t} k(\sigma, s).$$

2. Investigation of the Direct Problem

Apply the Fourier method to investigate the for direct problems (1)–(5). For this purpose

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, \quad (t, x, y) \in \Omega, \tag{7}$$

we look for a non-trivial particular solution of the equation in the form:

$$u(x, y, t) = Z(x, y)v(t). \tag{8}$$

Substituting this expression into the equation (7) and boundary conditions (4), (5) and separating the variables, we obtain the problem for finding eigenfunctions

$$\frac{\partial^2 Z}{\partial x^2} + \frac{\partial^2 Z}{\partial y^2} + \mu Z = 0, \quad (x, y) \in D, \tag{9}$$

$$\begin{aligned} Z_x(0, y) = Z_x(1, y), \quad Z(0, y) = 0, \quad 0 \leqslant y \leqslant 1, \\ Z(x, 0) = Z(x, 1), \quad Z_y(x, 1) = 0, \quad 0 \leqslant x \leqslant 1. \end{aligned} \tag{10}$$

The boundary value problem (9), (10) is not self-adjoint in the sense of the scalar product $(Z, W) = \int_0^1 \int_0^1 Z(x, y)W(x, y)\, dxdy$. Associated with it will be the following problem

$$\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} + \mu W = 0, \quad (x, y) \in D, \tag{11}$$

$$\begin{aligned} W(0, y) = W(1, y), \quad W_x(1, y) = 0, \quad 0 \leqslant y \leqslant 1, \\ W_y(x, 0) = W_y(x, 1), \quad W(x, 0) = 0, \quad 0 \leqslant x \leqslant 1. \end{aligned} \tag{12}$$

First we solve the problem (9), (10). To do this, we represent its solution in the following form

$$Z(x, y) = X(x)Y(y). \tag{13}$$

Substituting this expression into the equation (9) and using the boundary conditions (10), we obtain the following spectral problems

$$\begin{gathered} X''(x) + \lambda X(x) = 0, \quad 0 < x < 1, \\ X'(0) = X'(1), \quad X(0) = 0, \end{gathered} \tag{14}$$

$$\begin{gathered} Y''(y) + \gamma Y(y) = 0, \quad 0 < y < 1, \\ Y(0) = Y(1), \quad Y'(1) = 0, \end{gathered} \tag{15}$$

where $\mu = \lambda + \gamma$.

In work [18], it is shown that the solution of the problem (14) has a Riesz basis in $L_2(0,1)$, consisting of eigenfunctions and associated functions $X_0(x) = x$, $X_{2m}(x) = \sin(2\pi mx)$, $X_{2m-1}(x) = x\cos(2\pi mx)$, and eigenvalues $\lambda_m = (2\pi m)^2$, $m = 0, 1, 2$.

Biorthogonal with $\{X_m\}_0^\infty$ sequence of eigenfunctions and associated functions of the problem adjoint to the problem (14): $\overline{X}_0(x) = 2$, $\overline{X}_{2m}(x) = 4(1-x)\sin(2\pi mx)$, $\overline{X}_{2m-1}(x) = 4\cos(2\pi mx)$, $m = 1, 2, \ldots$, also forms a Riesz basis. The solution of the problem (15) is similar to obtain eigenfunctions and associated functions $Y_0(y) = 2$, $Y_{2k}(y) = \cos(2\pi ky)$, $Y_{2k-1}(y) = (1-y)\sin(2\pi ky)$, and eigenvalues $\gamma_k = (2\pi k)^2$, $k = 0, 1, 2, \ldots$

A biorthogonal sequence with $\{Y_k\}_0^\infty$ is similar, we obtain eigenfunctions and associated functions $\overline{Y}_0(y) = y$, $\overline{Y}_{2k}(y) = 4y\cos(2\pi ky)$, $\overline{Y}_{2k-1}(y) = 4\sin(2\pi ky)$. The system of eigenfunctions and associated functions for the problem (9), (10) will be redesignated as follows

$$Z_{i,j}(x,y) = X_i(x)Y_j(y), \quad i = \{0, 2m-1, 2m\}, \; j = \{0, 2k-1, 2k\}, \tag{16}$$

where $k, m = 1, 2, \ldots$

The eigenfunctions and associated functions of the adjoint problem (11), (12) have the form

$$W_{i,j}(x,y) = \overline{X}_i(x)\overline{Y}_j(y), \quad i = \{0, 2m-1, 2m\}, \; j = \{0, 2k-1, 2k\}. \tag{17}$$

Note that the systems of sequences of functions (16), (17) form admissible numbers m, k, l, p, the following relations hold: $(Z_{m,k}, W_{l,p}) = 1$ if $m = l$, $k = p$; otherwise $(Z_{m,k}, W_{l,p}) = 0$.

According to (8), particular solutions of the problem (9), (10) can be represented as an expansion in the series

$$\begin{aligned} u(x,y,t) = {} & Z_{0,0}(x,y)v_{0,0}(t) + \sum_{k=1}^{\infty} Z_{0,2k-1}(x,y)v_{0,2k-1}(t) + \sum_{k=1}^{\infty} Z_{0,2k}(x,y)v_{0,2k}(t) \\ & + \sum_{m=1}^{\infty} Z_{2m-1,0}(x,y)v_{2m-1,0}(t) + \sum_{m,k=1}^{\infty} Z_{2m-1,2k-1}(x,y)v_{2m-1,2k-1}(t) \\ & + \sum_{m,k=1}^{\infty} Z_{2m-1,2k}(x,y)v_{2m-1,2k}(t) + \sum_{m=1}^{\infty} Z_{2m,0}(x,y)v_{2m,0}(t) \\ & + \sum_{m,k=1}^{\infty} Z_{2m,2k-1}(x,y)v_{2m,2k-1}(t) + \sum_{m,k=1}^{\infty} Z_{2m,2k}(x,y)v_{2m,2k}(t). \end{aligned} \tag{18}$$

Coefficients $v_{0,0}(t)$, $v_{0,2k-1}(t)$, $v_{0,2k}(t)$, $v_{2m-1,0}(t)$, $v_{2m-1,2k-1}(t)$, $v_{2m-1,2k}(t)$, $v_{2m,0}(t)$, $v_{2m,2k-1}(t)$, $v_{2m,2k}(t)$ for $m, k \geqslant 1$ are to be found by making use of the orthogonality of the eigenfunctions. Namely, we multiply (1) by the eigenfunctions (16) and integrate over $((0,1) \times (0,1))$. Recall that the scalar product in $L_2((0,1) \times (0,1))$ is defined by

$(f,g)=\int_0^1\int_0^1 f(x,y)g(x,y)\,dxdy$. Let us note the expansion coefficients of $f(x,y)$ and $g(x,y)$ in the eigenfunctions of (17) for $m,k\geqslant 1$ respectively by.

Note the expansion coefficients of the functions $f(x,y,t)$ and $\varphi_i(x,y)$, $i=1,2$, in terms of eigenfunctions (17), respectively, for

$$(f(x,y,t),W_{i,j}(x,y))=f_{i,j}(t),\quad i=\{0,2m-1,2m\}\,,\ j=\{0,2k-1,2k\}\,, \tag{19}$$

$$(\varphi_l(x,y),W_{i,j}(x,y))=\varphi_{i,j,l},\quad l=\{1,2\}\,,\ i=\{0,2m-1,2m\}\,,\ j=\{0,2k-1,2k\}\,. \tag{20}$$

We get with (1) and $(u(x,y,t),W_{0,0}(x,y))=v_{0,0}(t)$ and the first component (19), (20) can be written as

$$\begin{cases} v''_{0,0}(t)+q(t)v_{0,0}(t)=f_{0,0}(t),\\ v_{0,0}(t)\big|_{t=0}=\varphi_{0,0,1},\ v'_{0,0}(t)\big|_{t=0}=\varphi_{0,0,2}. \end{cases} \tag{21}$$

We obtain the following inhomogeneous Cauchy problems for a second-order differential equation for arbitrary $m,k=1,2,\ldots$

$$\begin{cases} v''_{0,2k-1}(t)+\gamma_k v_{0,2k-1}+q(t)v_{0,2k-1}(t)=f_{0,2k-1}(t),\\ v_{0,2k-1}(t)\big|_{t=0}=\varphi_{0,2k-1,1},\ v'_{0,2k-1}(t)\big|_{t=0}=\varphi_{0,2k-1,2}, \end{cases} \tag{22}$$

$$\begin{cases} v''_{0,2k}(t)+\gamma_k v_{0,2k}(t)+2\sqrt{\gamma_k}\,v_{0,2k-1}(t)+q(t)v_{0,2k}(t)=f_{0,2k}(t),\\ v_{0,2k}(t)\big|_{t=0}=\varphi_{0,2k,1},\ v'_{0,2k}(t)\big|_{t=0}=\varphi_{0,2k,2}, \end{cases} \tag{23}$$

$$\begin{cases} v''_{2m-1,0}(t)+\lambda_m v_{2m-1,0}+q(t)v_{2m-1,0}(t)=f_{2m-1,0}(t),\\ v_{2m-1,0}(t)\big|_{t=0}=\varphi_{2m-1,0,1},\ v'_{2m-1,0}(t)\big|_{t=0}=\varphi_{2m-1,0,2}, \end{cases} \tag{24}$$

$$\begin{cases} v''_{2m-1,2k-1}(t)+\mu_{mk}v_{2m-1,2k-1}+q(t)v_{2m-1,2k-1}(t)=f_{2m-1,2k-1}(t),\\ v_{2m-1,2k-1}(t)\big|_{t=0}=\varphi_{2m-1,2k-1,1},\ v'_{2m-1,2k-1}(t)\big|_{t=0}=\varphi_{2m-1,2k-1,2}, \end{cases} \tag{25}$$

$$\begin{cases} v''_{2m-1,2k}(t)+\mu_{mk}v_{2m-1,2k}+2\sqrt{\gamma_k}\,v_{2m-1,2k-1}+q(t)v_{2m-1,2k}(t)=f_{2m-1,2k}(t),\\ v_{2m-1,2k}(t)\big|_{t=0}=\varphi_{2m-1,2k,1},\ v'_{2m-1,2k}(t)\big|_{t=0}=\varphi_{2m-1,2k,2}, \end{cases} \tag{26}$$

$$\begin{cases} v''_{2m,0}(t)+\lambda_m v_{2m,0}+2\sqrt{\lambda_m}\,v_{2m-1,0}+q(t)v_{2m,0}(t)=f_{2m,0}(t),\\ v_{2m,0}(t)\big|_{t=0}=\varphi_{2m,0,1},\ v'_{2m,0}(t)\big|_{t=0}=\varphi_{2m,0,2}, \end{cases} \tag{27}$$

$$\begin{cases} v''_{2m,2k-1}(t)+\mu_{mk}v_{2m,2k-1}+2\sqrt{\lambda_m}\,v_{2m-1,2k-1}+q(t)v_{2m,2k-1}(t)=f_{2m,2k-1}(t),\\ v_{2m,2k-1}(t)\big|_{t=0}=\varphi_{2m,2k-1,1},\ v'_{2m,2k-1}(t)\big|_{t=0}=\varphi_{2m,2k-1,2}, \end{cases} \tag{28}$$

$$\begin{cases} v''_{2m,2k}(t)+\mu_{mk}v_{2m,2k}+2\sqrt{\lambda_m}v_{2m-1,2k}+2\sqrt{\gamma_k}v_{2m,2k-1}+q(t)v_{2m,2k}(t)=f_{2m,2k}(t),\\ v_{2m,2k}(t)\big|_{t=0}=\varphi_{2m,2k,1},\ v'_{2m,2k}(t)\big|_{t=0}=\varphi_{2m,2k,2}. \end{cases} \tag{29}$$

The problem (21) is equivalent $C[0,T]$ to the Volterra integral equation of the second kind

$$v_{0,0}(t)=\int_0^t (t-\tau)\big[f_{0,0}(\tau)-q(\tau)v_{0,0}(\tau)\big]\,d\tau+\varphi_{0,0,2}t+\varphi_{0,0,1}. \tag{30}$$

Lemma 2. *The following estimates are valid*

$$|v_{0,0}(t)| \leqslant \left(|\varphi_{0,0,1}| + |\varphi_{0,0,2}|\,T + \frac{\|f_{0,0}\|_{C[0,T]}\,T^2}{2} \right) \exp\left\{ \frac{\|q\|_{C[0,T]}T^2}{2} \right\} =: \Psi_{0,0}(T),$$
$$\left|v''_{0,0}(t)\right| \leqslant \left\|f_{0,0}\right\|_{C[0,T]} + \left\|q\right\|_{C[0,T]}\Psi_{0,0}(T) =: \Upsilon_{0,0}(T). \tag{31}$$

$\lhd$ Obviously, the original inequality implies

$$|v_{0,0}(t)| \leqslant |\varphi_{0,0,1}| + |\varphi_{0,0,2}|\,t + \frac{\|f_{0,0}\|_{C[0,T]}\,t^2}{2} + \|q\|_{C[0,T]} \int_0^t (t-\tau)\,|v_{0,0}(\tau)|\,d\tau,$$

from Lemma 1, we get

$$|v_{0,0}(t)| \leqslant \left(|\varphi_{0,0,1}| + |\varphi_{0,0,2}|\,T + \frac{\|f_{0,0}\|_{C[0,T]}T^2}{2} \right) \exp\left\{ \frac{\|q\|_{C[0,T]}T^2}{2} \right\}.$$

From the last inequality we obtain the estimates of the Lemma 2 for any $t \in [0,T]$. The lemma is proven. $\rhd$

Solutions to the problem (22) have the form

$$v_{0,2k-1}(t) = \frac{1}{\sqrt{\gamma_k}} \int_0^t \sin\sqrt{\gamma_k}\,(t-\tau)\big[f_{0,2k-1}(\tau) - q(\tau)u_{0,2k-1}(\tau)\big]\,d\tau$$
$$+\,\varphi_{0,2k-1,1}\cos\sqrt{\gamma_k}\,t + \frac{\varphi_{0,2k-1,2}}{\sqrt{\gamma_k}}\sin\sqrt{\gamma_k}\,t. \tag{32}$$

The following lemma is proved in the same way as Lemma 2.

Lemma 3. *Estimates hold:*

$$\left|v_{0,2k-1}(t)\right| \leqslant |\varphi_{0,2k-1,1}|\cos\sqrt{\gamma_k}\,t + \frac{|\varphi_{0,2k-1,2}|\sin\sqrt{\gamma_k}\,t}{\sqrt{\gamma_k}}$$
$$+\,\frac{\|f_{0,2k-1}\|_{C[0,T]}t}{\gamma_k} + \frac{\|q\|_{C[0,T]}}{\sqrt{\gamma_k}} \int_0^t \sin\sqrt{\gamma_k}\,(t-\tau)\,v_{0,0}(\tau)\,d\tau$$
$$\leqslant \left(|\varphi_{0,2k-1,1}| + \frac{|\varphi_{0,2k-1,2}|}{\sqrt{\gamma_k}} + \frac{\|f_{0,2k-1}\|_{C[0,T]}T}{\gamma_k} \right) \exp\left\{ \frac{\|q\|_{C[0,T]}T}{\gamma_k} \right\} =: \Psi_{0,2k-1}(T), \tag{33}$$
$$\left|v''_{0,2k-1}(t)\right| \leqslant \left\|f_{0,2k-1}\right\|_{C[0,T]} + \big(\gamma_k + \|q\|_{C[0,T]}\big)\Psi_{0,2k-1}(T).$$

For the problem (23)–(29), we obtain the equivalent integral equations,

$$v_{0,2k}(t) = \frac{1}{\sqrt{\gamma_k}} \int_0^t \sin\sqrt{\gamma_k}\,(t-\tau)\,\big[f_{0,2k}(\tau) - q(\tau)v_{0,2k}(\tau)$$
$$-\,2\sqrt{\gamma_k}\,v_{0,2k-1}(\tau)\big]\,d\tau\varphi_{0,2k,1}\cos\sqrt{\gamma_k}\,t + \frac{\varphi_{0,2k,2}}{\sqrt{\gamma_k}}\sin\sqrt{\gamma_k}\,t, \tag{34}$$

$$v_{2m-1,0}(t) = \varphi_{2m-1,0,1}\cos\sqrt{\lambda_m}\,t + \frac{\varphi_{2m-1,0,2}}{\sqrt{\lambda_m}}\sin\sqrt{\lambda_m}\,t$$
$$+\,\frac{1}{\sqrt{\lambda_m}} \int_0^t \sin\sqrt{\lambda_m}(t-\tau)\,\big[f_{2m-1,0}(\tau) - q(\tau)v_{2m-1,0}(\tau)\big]\,d\tau, \tag{35}$$

$$v_{2m-1,2k-1}(t) = \varphi_{2m-1,2k-1,1} \cos \sqrt{\mu_{mk}}\, t + \frac{\varphi_{2m-1,2k-1,2}}{\sqrt{\mu_{mk}}} \sin \sqrt{\mu_{mk}}\, t$$
$$+ \frac{1}{\sqrt{\mu_{mk}}} \int\limits_0^t \sin \sqrt{\mu_{mk}}\,(t-\tau) \left[f_{2m-1,2k-1}(\tau) - q(\tau) v_{2m-1,2k-1}(\tau) \right] d\tau, \tag{36}$$

$$v_{2m-1,2k}(t) = \frac{1}{\sqrt{\mu_{mk}}} \int\limits_0^t \sin \sqrt{\mu_{mk}}\,(t-\tau) \big[f_{2m-1,2k}(\tau) - q(\tau) v_{2m-1,2k}(\tau)$$
$$- 2\sqrt{\gamma_k}\, v_{2m-1,2k-1}(\tau) \big] d\tau + \varphi_{2m-1,2k,1} \cos \sqrt{\mu_{mk}}\, t + \frac{\varphi_{2m-1,2k,2}}{\sqrt{\mu_{mk}}} \sin \sqrt{\mu_{mk}}\, t, \tag{37}$$

$$v_{2m,0}(t) = \frac{1}{\sqrt{\lambda_m}} \int\limits_0^t \sin \sqrt{\lambda_m}\,(t-\tau) \big[f_{2m,0}(\tau) - q(\tau) v_{2m,0}(\tau)$$
$$-2\sqrt{\lambda_m} v_{2m-1,0}(\tau) \big] d\tau + \varphi_{2m,0,1} \cos \sqrt{\lambda_m}\, t + \frac{\varphi_{2m,0,2}}{\sqrt{\lambda_m}} \sin \sqrt{\lambda_m}\, t, \tag{38}$$

$$v_{2m,2k-1}(t) = \frac{1}{\sqrt{\mu_{mk}}} \int\limits_0^t \sin \sqrt{\mu_{mk}}\,(t-\tau) \big[f_{2m,2k-1}(\tau) - q(\tau) v_{2m,2k-1}(\tau)$$
$$- 2\sqrt{\lambda_m} v_{2m-1,2k-1}(\tau) \big] d\tau + \varphi_{2m,2k-1,1} \cos \sqrt{\mu_{mk}}\, t + \frac{\varphi_{2m,2k-1,2}}{\sqrt{\mu_{mk}}} \sin \sqrt{\mu_{mk}}\, t, \tag{39}$$

$$v_{2m,2k}(t) = \frac{1}{\sqrt{\mu_{mk}}} \int\limits_0^t \sin \sqrt{\mu_{mk}}\,(t-\tau) \big[f_{2m,2k}(\tau) - q(\tau) v_{2m,2k}(\tau)$$
$$- 2\sqrt{\gamma_k}\, v_{2m,2k-1} - 2\sqrt{\lambda_m}\, v_{2m-1,2k} \big] d\tau + \varphi_{2m,2k,1} \cos \sqrt{\mu_{mk}}\, t + \frac{\varphi_{2m,2k,2}}{\sqrt{\mu_{mk}}} \sin \sqrt{\mu_{mk}}\, t. \tag{40}$$

The integral equations (34)–(40) have the following estimates:

$$\left| v_{0,2k}(t) \right| \leqslant \left(|\varphi_{0,2k,1}| + \frac{|\varphi_{0,2k,2}|}{\sqrt{\gamma_k}} + \frac{\|f_{0,2k}\|_{C[0,T]} T}{\gamma_k} + 2T\Psi_{0,2k-1}(T) \right) \exp\left\{ \frac{\|q\|_{C[0,T]} T}{\gamma_k} \right\} =: \Psi_{0,2k}(T),$$
$$\left| v''_{0,2k}(t) \right| \leqslant \left\| f_{0,2k} \right\|_{C[0,T]} + \left(\gamma_k + \|q\|_{C[0,T]} \right) \Psi_{0,2k} + 2\sqrt{\gamma_k}\, \Psi_{0,2k-1} =: \Upsilon_{0,2k}(T), \tag{41}$$

$$\left| v_{2m-1,0}(t) \right| \leqslant \left(|\varphi_{2m-1,0,1}| + \frac{|\varphi_{2m-1,0,2}|}{\sqrt{\lambda_m}} + \frac{\|f_{2m-1,0}\|_{C[0,T]} T}{\lambda_m} \right) \exp\left\{ \frac{\|q\|_{C[0,T]} T}{\lambda_m} \right\} =: \Psi_{2m-1,0}(T),$$
$$\left| v''_{2m-1,0}(t) \right| \leqslant \left\| f_{2m-1,0} \right\|_{C[0,T]} + \left(\lambda_m + \|q\|_{C[0,T]} \right) \Psi_{2m-1,0}(T) =: \Upsilon_{2m-1,0}(T), \tag{42}$$

$$\left| v_{2m-1,2k-1}(t) \right| \leqslant \Bigg(|\varphi_{2m-1,2k-1,1}| + \frac{|\varphi_{2m-1,2k-1,2}|}{\sqrt{\mu_{mk}}}$$
$$+ \frac{\|f_{2m-1,2k-1}\|_{C[0,T]} T}{\mu_{mk}} \Bigg) \exp\left\{ \frac{\|q\|_{C[0,T]} T}{\mu_{mk}} \right\} =: \Psi_{2m-1,2k-1}(T),$$

$$\left| v''_{2m-1,2k-1}(t) \right| \leqslant \left\| f_{2m-1,2k-1} \right\|_{C[0,T]} + \left(\mu_{mk} + \|q\|_{C[0,T]} \right) \Psi_{2m-1,2k-1}(T) =: \Upsilon_{2m-1,2k-1}(T), \tag{43}$$

$$\left|v_{2m-1,2k}(t)\right| \leqslant \left(|\varphi_{2m-1,2k,1}| + \frac{|\varphi_{2m-1,2k,2}|}{\sqrt{\mu_{mk}}} + \frac{\|f_{2m-1,2k}\|_{C[0,T]}T}{\mu_{mk}} \right.$$
$$\left. + 2T\sqrt{\frac{\gamma_k}{\mu_{mk}}}\,\Psi_{2m-1,2k-1}(T)\right)\exp\left\{\frac{\|q\|_{C[0,T]}T}{\mu_{mk}}\right\} =: \Psi_{2m-1,2k}(T),$$

$$\left|v''_{2m-1,2k}(t)\right| \leqslant \left(\mu_{mk} + \|q\|_{C[0,T]}\right)\Psi_{2m-1,2k}(T)$$
$$+ \left\|f_{2m-1,2k}\right\|_{C[0,T]} + 2\sqrt{\gamma_k}\,\Psi_{2m-1,2k-1}(T) =: \Upsilon_{2m-1,2k}(T), \quad (44)$$

$$\left|v_{2m,0}(t)\right| \leqslant \left(|\varphi_{2m,0,1}| + \frac{|\varphi_{2m,0,2}|}{\sqrt{\lambda_m}} + \frac{\|f_{2m,0}\|_{C[0,T]}T}{\lambda_m} \right.$$
$$\left. + 2T\Psi_{2m-1,0}(T)\right)\exp\left\{\frac{\|q\|_{C[0,T]}T}{\lambda_m}\right\} =: \Psi_{2m,0}(T),$$

$$\left|v''_{2m,0}(t)\right| \leqslant \left\|f_{2m,0}\right\|_{C[0,T]} + \left(\lambda_m + \|q\|_{C[0,T]}\right)\Psi_{2m,0}(T) + 2\sqrt{\lambda_m}\,\Psi_{2m-1,0}(T) =: \Upsilon_{2m,0}(T), \quad (45)$$

$$\left|v_{2m,2k-1}(t)\right| \leqslant \left(|\varphi_{2m,2k-1,1}| + \frac{|\varphi_{2m,2k-1,2}|}{\sqrt{\mu_{mk}}} + \frac{\|f_{2m,2k-1}\|_{C[0,T]}T}{\mu_{mk}} \right.$$
$$\left. + 2T\sqrt{\frac{\lambda_m}{\mu_{mk}}}\,\Psi_{2m-1,2k-1}(T)\right)\exp\left\{\frac{\|q\|_{C[0,T]}T}{\mu_{mk}}\right\} =: \Psi_{2m,2k-1}(T),$$

$$\left|v''_{2m,2k-1}(t)\right| \leqslant \left(\mu_{mk} + \|q\|_{C[0,T]}\right)\Psi_{2m,2k-1}(T)$$
$$+ \left\|f_{2m,2k-1}\right\|_{C[0,T]} + 2\sqrt{\lambda_m}\,\Psi_{2m-1,2k-1}(T) =: \Upsilon_{2m,2k-1}(T), \quad (46)$$

$$\left|v_{2m,2k}(t)\right| \leqslant \left(|\varphi_{2m,2k,1}| + \frac{|\varphi_{2m,2k,2}|}{\sqrt{\mu_{mk}}} + \frac{\|f_{2m,2k}\|_{C[0,T]}T}{\mu_{mk}} + 2T\sqrt{\frac{\lambda_m}{\mu_{mk}}}\,\Psi_{2m-1,2k}(T) \right.$$
$$\left. + 2T\sqrt{\frac{\gamma_k}{\mu_{mk}}}\,\Psi_{2m,2k-1}(T)\right)\exp\left\{\frac{\|q\|_{C[0,T]}T}{\mu_{mk}}\right\} =: \Psi_{2m,2k}(T),$$

$$\left|v''_{2m,2k}(t)\right| \leqslant \left(\mu_{mk} + \|q\|_{C[0,T]}\right)\Psi_{2m,2k}(T) + \left\|f_{2m,2k-1}\right\|_{C[0,T]}$$
$$+ 2\sqrt{\lambda_m}\,\Psi_{2m-1,2k}(T) + 2\sqrt{\gamma_k}\,\Psi_{2m,2k-1}(T) =: \Upsilon_{2m,2k}(T). \quad (47)$$

Solution to the problem (1)–(5) will be sought in the form (18) of the double Fourier series. Formally differentiating the series (18) term by term, we obtain the following series

$$\begin{aligned} u_{tt}(x,y,t) &= Z_{0,0}(x,y)v''_{0,0}(t) + \sum_{k=1}^{\infty} Z_{0,2k-1}(x,y)v''_{0,2k-1}(t) + \sum_{k=1}^{\infty} Z_{0,2k}(x,y)v''_{0,2k}(t) \\ &+ \sum_{m=1}^{\infty} Z_{2m-1,0}(x,y)v''_{2m-1,0}(t) + \sum_{m,k=1}^{\infty} Z_{2m-1,2k-1}(x,y)v''_{2m-1,2k-1}(t) \\ &+ \sum_{m,k=1}^{\infty} Z_{2m-1,2k}(x,y)v''_{2m-1,2k}(t) + \sum_{m=1}^{\infty} Z_{2m,0}(x,y)v''_{2m,0}(t) \\ &+ \sum_{m,k=1}^{\infty} Z_{2m,2k-1}(x,y)v''_{2m,2k-1}(t) + \sum_{m,k=1}^{\infty} Z_{2m,2k}(x,y)v''_{2m,2k}(t), \end{aligned} \quad (48)$$

$$u_{xx}(x,y,t) = -4\sum_{m=1}^{\infty}\sqrt{\lambda_m}\,v_{2m-1,0}(t)\sin\sqrt{\lambda_m}\,x - 2x\sum_{m=1}^{\infty}\lambda_m v_{2m-1,0}(t)\cos\sqrt{\lambda_m}\,x$$

$$-2(1-y)\sum_{m,k=1}^{\infty}\sqrt{\lambda_m}\,v_{2m-1,2k-1}(t)\sin\sqrt{\lambda_m}\,x\sin\sqrt{\gamma_k}\,y - x(1-y)$$

$$\times\sum_{m,k=1}^{\infty}\lambda_m v_{2m-1,2k-1}(t)\cos\sqrt{\lambda_m}x\sin\sqrt{\gamma_k}y - 2\sum_{m,k=1}^{\infty}\sqrt{\lambda_m}v_{2m-1,2k}(t)\sin\sqrt{\lambda_m}x\cos\sqrt{\gamma_k}y \quad (49)$$

$$-2\sum_{m=1}^{\infty}\lambda_m v_{2m,0}(t)\sin\sqrt{\lambda_m}\,x - x\sum_{m,k=1}^{\infty}\lambda_m v_{2m-1,2k}(t)\cos\sqrt{\lambda_m}\,x\cos\sqrt{\gamma_k}\,y$$

$$-\sum_{m,k=1}^{\infty}\lambda_m v_{2m,2k}(t)\sin\sqrt{\lambda_m}x\cos\sqrt{\gamma_k}y - (1-y)\sum_{m,k=1}^{\infty}\lambda_m v_{2m,2k-1}(t)\sin\sqrt{\lambda_m}x\sin\sqrt{\gamma_k}y,$$

$$u_{yy}(x,y,t) = -2x\sum_{k=1}^{\infty}\sqrt{\gamma_k}\,v_{0,2k-1}(t)\cos\sqrt{\gamma_k}\,y - x(1-y)\sum_{k=1}^{\infty}\gamma_k v_{0,2k-1}(t)\sin\sqrt{\gamma_k}\,y$$

$$-2x\sum_{m,k=1}^{\infty}\sqrt{\gamma_k}v_{2m-1,2k-1}(t)\cos\sqrt{\lambda_m}x\cos\sqrt{\gamma_k}y - x\sum_{k=1}^{\infty}\gamma_k v_{0,2k}(t)\cos\sqrt{\gamma_k}y - x(1-y)$$

$$\times\sum_{m,k=1}^{\infty}\gamma_k v_{2m-1,2k-1}(t)\cos\sqrt{\lambda_m}x\sin\sqrt{\gamma_k}y - x\sum_{m,k=1}^{\infty}\gamma_k v_{2m-1,2k}(t)\cos\sqrt{\lambda_m}x\cos\sqrt{\gamma_k}y \quad (50)$$

$$-2\sum_{m,k=1}^{\infty}\sqrt{\gamma_k}\,v_{2m,2k-1}(t)\sin\sqrt{\lambda_m}\,x\cos\sqrt{\gamma_k}\,y - (1-y)\sum_{m,k=1}^{\infty}\gamma_k v_{2m,2k-1}(t)$$

$$\times\sin\sqrt{\lambda_m}\,x\sin\sqrt{\gamma_k}y - \sum_{m,k=1}^{\infty}\gamma_k v_{2m,2k}(t)\sin\sqrt{\lambda_m}\,x\cos\sqrt{\gamma_k}y.$$

The series (18), (48)–(50) due to the estimates (31),(33),(41)–(47) respectively for any $(x,y,t)\in\overline{\Omega}$ are majorized by the exspertions

$$2\Psi_{0,0}(T) + \sum_{k=1}^{\infty}\Psi_{0,2k-1}(T) + \sum_{k=1}^{\infty}\Psi_{0,2k}(T) + 2\sum_{m=1}^{\infty}\Psi_{2m-1,0}(T) + \sum_{m,k=1}^{\infty}\Psi_{2m-1,2k-1}(T)$$
$$+\sum_{m,k=1}^{\infty}\Psi_{2m-1,2k}(T) + 2\sum_{m=1}^{\infty}\Psi_{2m,0}(T) + \sum_{m,k=1}^{\infty}\Psi_{2m,2k-1}(T) + \sum_{m,k=1}^{\infty}\Psi_{2m,2k}(T), \quad (51)$$

$$2\Upsilon_{0,0}(T) + \sum_{k=1}^{\infty}\Upsilon_{0,2k-1}(T) + \sum_{k=1}^{\infty}\Upsilon_{0,2k}(T) + 2\sum_{m=1}^{\infty}\Upsilon_{2m-1,0}(T) + \sum_{m,k=1}^{\infty}\Upsilon_{2m-1,2k-1}(T)$$
$$+\sum_{m,k=1}^{\infty}\Upsilon_{2m-1,2k}(T) + 2\sum_{m=1}^{\infty}\Upsilon_{2m,0}(T) + \sum_{m,k=1}^{\infty}\Upsilon_{2m,2k-1}(T) + \sum_{m,k=1}^{\infty}\Upsilon_{2m,2k}(T), \quad (52)$$

$$4\sum_{m=1}^{\infty}\sqrt{\lambda_m}\,\Psi_{2m-1,0}(T)+2\sum_{m=1}^{\infty}\lambda_m\Psi_{2m-1,0}(T)+2\sum_{m,k=1}^{\infty}\sqrt{\lambda_m}\,\Psi_{2m-1,2k-1}(T)$$
$$+\sum_{m,k=1}^{\infty}\lambda_m\Psi_{2m-1,2k-1}(T)+2\sum_{m,k=1}^{\infty}\sqrt{\lambda_m}\,\Psi_{2m-1,2k}(T)+2\sum_{m=1}^{\infty}\lambda_m\Psi_{2m,0}(T) \tag{53}$$
$$+\sum_{m,k=1}^{\infty}\lambda_m\Psi_{2m-1,2k}(T)+\sum_{m,k=1}^{\infty}\lambda_m\Psi_{2m,2k-1}(T)+\sum_{m,k=1}^{\infty}\lambda_m\Psi_{2m,2k}(t),$$

$$2\sum_{k=1}^{\infty}\sqrt{\gamma_k}\,\Psi_{0,2k-1}(T)+\sum_{k=1}^{\infty}\gamma_k\Psi_{0,2k-1}(T)+\sum_{k=1}^{\infty}\gamma_k\Psi_{0,2k}(T)$$
$$+2\sum_{m,k=1}^{\infty}\sqrt{\gamma_k}\,\Psi_{2m-1,2k-1}(T)+\sum_{m,k=1}^{\infty}\gamma_k\Psi_{2m-1,2k-1}(T)+\sum_{m,k=1}^{\infty}\gamma_k\Psi_{2m-1,2k}(t) \tag{54}$$
$$+2\sum_{m,k=1}^{\infty}\sqrt{\gamma_k}\,\Psi_{2m,2k-1}(T)+\sum_{m,k=1}^{\infty}\gamma_k\Psi_{2m,2k-1}(T)+\sum_{m,k=1}^{\infty}\gamma_k\Psi_{2m,2k}(T).$$

Let us present the following lemma so that all the series $\sum_{m,k=1}^{\infty}\Psi_{m,k}(T)$, $\sum_{m,k=1}^{\infty}\Upsilon_{m,k}(T)$ the following assertotion the valid.

Lemma 4. *Let* $\varphi_1(x,y)\in C^4(\overline{D}\,),\varphi_2(x,y)\in C^3(\overline{D}\,)$ *and* $f(x,y,t)\in C^{2,0}_{xy,t}(\overline{\Omega}\,)$. *Besides, the following equalities hold*

$$\varphi_1(0,y)=\varphi_{1xx}(0,y)=\varphi_1(1,y)=\varphi_{1xx}(1,y),\quad 0\leqslant y\leqslant 1,$$
$$\varphi_1(x,0)=\varphi_{1yy}(x,0)=\varphi_1(x,1)=\varphi_{1yy}(x,1),\quad 0\leqslant x\leqslant 1,$$
$$\varphi_2(0,y)=\varphi_{2xx}(0,y)=\varphi_2(1,y)=\varphi_{2xx}(1,y),\quad 0\leqslant y\leqslant 1,$$
$$\varphi_2(x,0)=\varphi_{2yy}(x,0)=\varphi_2(x,1)=\varphi_{2yy}(x,1),\quad 0\leqslant x\leqslant 1,$$
$$f(0,y,t)=f(1,y,t),\quad (y,t)\in[0,1]\times[0,T],$$
$$f(x,0,t)=f(x,1,t),\quad (x,t)\in[0,1]\times[0,T].$$

Then the numerical series in (51)–(54) *are converge.*

$\lhd$ Integrating by parts $\varphi_{l,n}$ four times over x, taking into account the conditions of the lemma, we obtain

$$\lambda_l^2\varphi_{l,n}=\iint\limits_{\Omega}\varphi^{(4,0)}_{xxx}(x,y)\,W_{l,n}\,dxdy=\varphi^{(4,0)}_{l,n}.$$

Integrating by parts $\varphi_{l,n}$ three times over x and once over y (twice over x, twice over y, etc.), using the conditions of the lemma, we have

$$\varphi^{(3,1)}_{l,n}=\sqrt{\lambda_l^3}\sqrt{\gamma_n}\,\varphi_{l,n},\quad \varphi^{(2,2)}_{ln}=\lambda_l\gamma_n\varphi_{l,n},\quad \varphi^{(1,3)}_{l,n}=\sqrt{\lambda_l}\sqrt{\gamma_n^3}\,\varphi_{l,n},\quad \varphi^{(0,4)}_{l,n}=\gamma_n^2\varphi_{l,n}.$$

From here we get the following representation for $|\varphi_{l,n,1}|$, $l=0,2m-1,2m$, $n=0,2k-1,2k$,

$$\begin{aligned}|\varphi_{l,n,1}|&=\frac{|\varphi^{(4,0)}_{l,n,1}|+4|\varphi^{(3,1)}_{l,n,1}|+6|\varphi^{(2,2)}_{l,n,1}|+4|\varphi^{(1,3)}_{l,n,1}|+|\varphi^{(0,4)}_{l,n,1}|}{(2\pi)^2\left(\sqrt{l}+\sqrt{n}\right)^4}\\&=\frac{1}{(2\pi)^2\left(\sqrt{l}+\sqrt{n}\right)^4}\sum_{i+j=4}\binom{4}{j}\left|\varphi^{(i,j)}_{ln,1}\right|,\end{aligned} \tag{55}$$

where

$$\varphi_{l,n,1}^{(i,j)} = \iint\limits_{\Omega} \left(\frac{\partial^4 \varphi_1(x,y)}{\partial x^i \partial y^j} \right) W_{l,n}(x,y)\, dxdy, \quad i+j=4.$$

Having done a similar procedure for $|\varphi_{l,n,2}|$, $|f_{l,n}(t)|$ we have

$$|\varphi_{l,n,2}| = \frac{|\varphi_{l,n,2}^{(3,0)}| + 3|\varphi_{l,n,2}^{(2,1)}| + 3|\varphi_{l,n,2}^{(1,2)}| + |\varphi_{l,n,2}^{(0,3)}|}{(\sqrt{2}\pi)^3(\sqrt{l}+\sqrt{n})^3} = \frac{1}{(\sqrt{2}\pi)^3(\sqrt{l}+\sqrt{n})^3} \sum_{i+j=3} \binom{3}{j} \left|\varphi_{l,n,2}^{(i,j)}\right|, \quad (56)$$

$$|f_{l,n}(t)| = \frac{|f_{l,n}^{(2,0)}(t)| + 2|f_{l,n}^{(1,1)}(t)| + |f_{l,n}^{(0,2)}(t)|}{(\sqrt{2}\pi)^2(\sqrt{l}+\sqrt{n})^2} = \frac{1}{(\sqrt{2}\pi)^2(\sqrt{l}+\sqrt{n})^2} \sum_{i+j=2} \binom{2}{j} \left|f_{l,n}^{(i,j)}(t)\right|, \quad (57)$$

where

$$\varphi_{l,n,2}^{(i,j)} = \iint\limits_{\Omega} \left(\frac{\partial^3 \varphi_2(x,y)}{\partial x^i \partial y^j} \right) W_{l,n}(x,y)\, dxdy, \quad i+j=3,$$

$$f_{l,n}^{(i,j)} = \iint\limits_{\Omega} \left(\frac{\partial^2 f(x,y,t)}{\partial x^i \partial y^j} \right) W_{l,n}(x,y)\, dxdy, \quad i+j=2,$$

From the formulas (55)–(57), by Bessel's inequality, we obtain

$$\sum_{l,n}^{\infty} \left|\varphi_{l,n,1}^{(i,j)}\right|^2 \leqslant \iint\limits_{\Omega} \left(\frac{\partial^4 \varphi_1(x,y)}{\partial x^i \partial y^j} \right)^2 dxdy, \quad i+j=4, \quad (58)$$

$$\sum_{l,n}^{\infty} \left|\varphi_{l,n,2}^{(i,j)}\right|^2 \leqslant \iint\limits_{\Omega} \left(\frac{\partial^3 \varphi_2(x,y)}{\partial x^i \partial y^j} \right)^2 dxdy, \quad i+j=3, \quad (59)$$

$$\sum_{l,n}^{\infty} \left|f_{l,n}^{(i,j)}\right|^2 \leqslant \int\limits_{\Omega} \left(\frac{\partial^2 f(x,y)}{\partial x^i \partial y^j} \right)^2 dxdy, \quad i+j=2. \quad (60)$$

where $l = \{0, 2m-1, 2m\}$, $n = \{0, 2k-1, 2k\}$, $m, k = 1, 2, 3, \ldots$

The relations (55)–(60) imply the convergence of the series (51)–(54). Therefore, the series (18), (48)–(50) converge uniformly. $\rhd$

Thus, we have proved the following theorem.

Theorem 1. *Let $q(t) \in C[0,T]$ and if the functions $\varphi_1(x,y)$, $\varphi_2(x,y)$ and $f(x,y,t)$ satisfy the conditions of the Lemma* 4, *then there exists a unique solution to the problem* (1)–(5).

Let us estimate the norm of the difference between the solution of the original integral equation (30), (32), (34)–(40) and the solution of this equation with perturbed functions $\widetilde{q}, \widetilde{\varphi}_{mk}, \widetilde{\varphi}_{mk}$ and $\widetilde{f}_{mk}$. Let $\widetilde{v}_{mk}(t)$ be the solution of the integral equation (30), (32), (34)–(40) corresponding to the functions $\widetilde{q}$, $\widetilde{\varphi}_{mk}$, $\widetilde{\varphi}_{mk}$, $\widetilde{f}_{mk}$, i. e.,

$$\widetilde{v}_{0,0}(t) = \widetilde{\varphi}_{0,0,1} + \widetilde{\varphi}_{0,0,2} t + \int\limits_0^t (t-\tau)\big[\widetilde{f}_{0,0}(\tau) - \widetilde{q}(\tau)\widetilde{v}_{0,0}(\tau)\big] d\tau. \quad (61)$$

Composing the difference $v_{0,0}(t) - \widetilde{v}_{0,0}(t)$ using equations (30), (32), (34)–(40) and introducing the notation $\widehat{v}_{0,0}(t) = v_{0,0}(t) - \widetilde{v}_{0,0}(t)$, $\widehat{q}_{0,0}(t) = q_{0,0}(t) - \widetilde{q}_{0,0}(t)$,

$\widehat{\varphi}_{0,0,1}(t) = \varphi_{0,0,1}(t) - \widetilde{\varphi}_{0,0,1}(t)$, $\widehat{\varphi}_{0,0,2}(t) = \varphi_{0,0,2}(t) - \widetilde{\varphi}_{0,0,2}(t)$, $\widehat{f}_{0,0}(t) = f_{0,0,1}(t) - \widetilde{f}_{0,0,1}(t)$ we get the integral equation

$$\widehat{v}_{0,0}(t) = \widehat{\varphi}_{0,0,1} + \widehat{\varphi}_{0,0,2}t + \int\limits_0^t (t-\tau)\left[\widehat{f}_{0,0}(\tau) - \widehat{q}(\tau)v_{0,0}(\tau)\right] d\tau - \int_0^t (t-\tau)\,\widetilde{q}(\tau)\,\widehat{v}_{0,0}(\tau)\,d\tau. \quad (62)$$

Hence we derive the following inequality

$$\begin{gathered}\left|\widehat{v}_{0,0}(t)\right| \leqslant \left[\left|\widehat{\varphi}_{0,0,1}\right| + \left|\widehat{\varphi}_{0,0,2}\right|T + \frac{\|\widehat{f}_{0,0}\|_{C[0,T]}T^2}{2} + \frac{\|\widehat{q}\,\|_{C[0,T]}T^2}{2}\right. \\ \left.\times\left(|\varphi_{0,0,1}| + |\varphi_{0,0,2}|\,T + \frac{\|f_{0,0}\|_{C[0,T]}T^2}{2}\right)\exp\left\{\frac{\|q\|_{C[0,T]}T^2}{2}\right\}\right]\exp\left\{\frac{\|\widetilde{q}\|_{C[0,T]}T^2}{2}\right\}.\end{gathered} \quad (63)$$

Similarly, for another functions $\widehat{v}_{0,2k-1}(t)$, $\widehat{v}_{0,2k}(t)$, $\widehat{v}_{2m-1,0}(t)$, $\widehat{v}_{2m-1,2k-1}(t)$, $\widehat{v}_{2m-1,2k}(t)$, $\widehat{v}_{2m,0}(t)$, $\widehat{v}_{2m,2k-1}(t)$, $\widehat{v}_{2m,2k}(t)$ we get the following estimates

$$\begin{gathered}\left|\widehat{v}_{0,2k-1}(t)\right| \leqslant \left(\left|\widehat{\varphi}_{0,2k-1,1}\right| + \frac{|\widehat{\varphi}_{0,2k-1,2}|}{\sqrt{\gamma_k}} + \frac{\|\widehat{f}_{0,2k-1}\|_{C[0,T]}T}{\gamma_k}\right. \\ \left.+\frac{\|\widehat{q}\,\|_{C[0,T]}T}{\gamma_k}\,\Psi_{0,2k-1}(T)\right)\exp\left\{\frac{\|\widetilde{q}\,\|_{C[0,T]}T}{\gamma_k}\right\} = \widehat{\Psi}_{0,2k-1}(T),\end{gathered} \quad (64)$$

$$\begin{gathered}\left|\widehat{v}_{0,2k}(t)\right| \leqslant \left(\left|\widehat{\varphi}_{0,2k,1}\right| + \frac{|\widehat{\varphi}_{0,2k,2}|}{\sqrt{\gamma_k}} + \frac{\|\widehat{f}_{0,2k-1}\|_{C[0,T]}T}{\gamma_k}\right. \\ \left.+\frac{\|\widehat{q}\,\|_{C[0,T]}T}{\gamma_k}\,\Psi_{0,2k}(T) + 2T\widehat{\Psi}_{0,2k-1}\right)\exp\left\{\frac{\|\widetilde{q}\,\|_{C[0,T]}T}{\gamma_k}\right\} = \widehat{\Psi}_{0,2k}(T),\end{gathered} \quad (65)$$

$$\begin{gathered}\left|\widehat{v}_{2m-1,0}(t)\right| \leqslant \left(\left|\widehat{\varphi}_{2m-1,0,1}\right| + \frac{|\widehat{\varphi}_{2m-1,0,2}|}{\sqrt{\lambda_m}} + \frac{\|\widehat{f}_{2m-1}\|_{C[0,T]}T}{\lambda_m}\right. \\ \left.+\frac{\|\widehat{q}\,\|_{C[0,T]}T}{\lambda_m}\,\Psi_{2m-1,0}(T)\right)\exp\left\{\frac{\|\widetilde{q}\,\|_{C[0,T]}T}{\lambda_m}\right\} = \widehat{\Psi}_{2m-1,0}(T),\end{gathered} \quad (66)$$

$$\begin{gathered}\left|\widehat{v}_{2m-1,2k-1}(t)\right| \leqslant \left(\left|\widehat{\varphi}_{2m-1,2k-1,1}\right| + \frac{|\widehat{\varphi}_{2m-1,2k-1,2}|}{\sqrt{\mu_{mk}}} + \frac{\|\widehat{f}_{2m-1,2k-1}\|_{C[0,T]}T}{\mu_{mk}}\right. \\ \left.+\frac{\|\widehat{q}\,\|_{C[0,T]}T}{\mu_{mk}}\,\Psi_{2m-1,2k-1}(T)\right)\exp\left\{\frac{\|\widetilde{q}\,\|_{C[0,T]}T}{\mu_{mk}}\right\} = \widehat{\Psi}_{2m-1,2k-1}(T),\end{gathered} \quad (67)$$

$$\begin{gathered}\left|\widehat{v}_{2m-1,2k}(t)\right| \leqslant \left(\left|\widehat{\varphi}_{2m-1,2k,1}\right| + \frac{|\widehat{\varphi}_{2m-1,2k,2}|}{\sqrt{\mu_{mk}}} + \frac{\|\widehat{f}_{2m-1,2k}\|_{C[0,T]}T}{\mu_{mk}} + \frac{\|\widehat{q}\,\|_{C[0,T]}T}{\mu_{mk}}\right. \\ \left.\times\,\Psi_{2m-1,2k}(T) + 2T\sqrt{\frac{\gamma_k}{\mu_{mk}}}\,\widehat{\Psi}_{2m-1,2k-1}(T)\right)\exp\left\{\frac{\|\widetilde{q}\,\|_{C[0,T]}T}{\mu_{mk}}\right\} = \widehat{\Psi}_{2m-1,2k}(T),\end{gathered} \quad (68)$$

$$\begin{gathered}\left|\widehat{v}_{2m,0}(t)\right| \leqslant \left(\left|\widehat{\varphi}_{2m,0,1}\right| + \frac{|\widehat{\varphi}_{2m,0,2}|}{\sqrt{\lambda_m}} + \frac{\|\widehat{f}_{2m,0}\|_{C[0,T]}T}{\lambda_m} + 2T\widehat{\Psi}_{2m-1,0}(T)\right. \\ \left.+\frac{\|\widehat{q}\,\|_{C[0,T]}T}{\lambda_m}\,\Psi_{2m,0}(T)\right)\exp\left\{\frac{\|\widetilde{q}\,\|_{C[0,T]}T}{\lambda_m}\right\} = \widehat{\Psi}_{2m,0}(T),\end{gathered} \quad (69)$$

$$\left|\widehat{v}_{2m,2k-1}(t)\right| \leqslant \Bigg(\left|\widehat{\varphi}_{2m,2k-1,1}\right| + \frac{|\widehat{\varphi}_{2m,2k-1,2}|}{\sqrt{\mu_{mk}}} + \frac{\|\widehat{f}_{2m,2k-1}\|_{C[0,T]}T}{\mu_{mk}} + \frac{\|\widehat{q}\,\|_{C[0,T]}T}{\mu_{mk}} \times \Psi_{2m,2k-1}(T) + 2T\sqrt{\frac{\lambda_m}{\mu_{mk}}}\,\widehat{\Psi}_{2m-1,2k-1}(T)\Bigg) \exp\left\{\frac{\|\widetilde{q}\,\|_{C[0,T]}T}{\mu_{mk}}\right\} = \widehat{\Psi}_{2m,2k-1}(T), \tag{70}$$

$$\left|\widehat{v}_{2m,2k}(t)\right| \leqslant \Bigg(\left|\widehat{\varphi}_{2m,2k,1}\right| + \frac{|\widehat{\varphi}_{2m,2k,2}|}{\sqrt{\mu_{mk}}} + 2T\sqrt{\frac{\lambda_m}{\mu_{mk}}}\widehat{\Psi}_{2m-1,2k}(T) + 2T\sqrt{\frac{\gamma_k}{\mu_{mk}}}\widehat{\Psi}_{2m,2k-1}(T) + \frac{\|\widehat{f}_{2m,2k}\|_{C[0,T]}T}{\mu_{mk}} + \frac{\|\widehat{q}\,\|_{C[0,T]}T}{\mu_{mk}}\Psi_{2m,2k}(T)\Bigg) \exp\left\{\frac{\|\widetilde{q}\,\|_{C[0,T]}T}{\mu_{mk}}\right\} = \widehat{\Psi}_{2m,2k}(T). \tag{71}$$

Indeed, the expressions (59)–(71) are stability estimates for the solutions to the problem (21)–(29). The uniqueness of these solutions follow from (59)–(71).

3. Study of the Inverse Problem (1)–(6)

The following assertion is main result in this poper:

Theorem 2. *Let the conditions of Lemma* 4 *and* $h(t) \in C^2[0,T]$, $|h(t)| \geqslant h_0 > 0$, *be satisfied and*

$$w(1,y) = w(0,y) = 0, \quad w_x(1,y) = 0, \quad y \in [0,1],$$
$$w(x,0) = 0, \quad w_y(x,1) = w_y(x,0) = 0, \quad x \in [0,1].$$

Then the exists $T^* \in (0,T)$ *so that the inverse problem* (1)–(6) *has unique solution* $q(t) \in C[0,T^*]$.

$\lhd$ Let us now proceed to constructing a solution to the inverse problem. Multiplying (1) by $w(x,y)$, integrating over x, y in $[0,1]\times[0,1]$, we get

$$\int_0^1\int_0^1 w(x,y)\{u_{tt} - \Delta u + q(t)u(x,y,t)\}\,dxdy = \int_0^1\int_0^1 w(x,y)f(x,y,t)\,dxdy.$$

Integrating by part second termo on the left side of this equation, twice over x and y,

$$h''(t) + q(t)h(t) - \int_0^1\int_0^1 \Delta w u(x,y,t)\,dxdy = \int_0^1\int_0^1 w(x,y)f(x,y,t)\,dxdy,$$

taking into account (18), this equations gives in view of conditions of Theorem 2 and using additional we have condition (6).

$$q(t) = \frac{1}{h(t)}\left(\int_0^1\int_0^1 w(x,y)f(x,y,t)\,dxdy - h''(t)\right) + \frac{1}{h(t)}\int_0^1\int_0^1 \Delta w\Bigg(Z_{0,0}(x,y)v_{0,0}(t) + \sum_{k=1}^{\infty} Z_{0,2k-1}(x,y)v_{0,2k-1}(t) + \sum_{k=1}^{\infty} Z_{0,2k}(x,y)v_{0,2k}(t) + \sum_{m=1}^{\infty} Z_{2m-1,0}(x,y)v_{2m-1,0}(t) + \sum_{m,k=1}^{\infty} Z_{2m-1,2k-1}(x,y)v_{2m-1,2k-1}(t) + \sum_{m,k=1}^{\infty} Z_{2m-1,2k}(x,y)v_{2m-1,2k}(t) + \sum_{m=1}^{\infty} Z_{2m,0}(x,y)v_{2m,0}(t) + \sum_{m,k=1}^{\infty} Z_{2m,2k-1}(x,y)v_{2m,2k-1}(t) + \sum_{m,k=1}^{\infty} Z_{2m,2k}(x,y)v_{2m,2k}(t)\Bigg)dxdy.$$

The function $v_{mk}(t)$ depends on $q(t)$, i. e., $u_{mk}(t;q)$. After a simple transformation, we obtain the following integral equation for determining $q(t)$:

$$q(t) = q_0(t) + \frac{1}{h(t)}\Bigg(w_{0,0}v_{0,0}(t;q) + \sum_{k=1}^{\infty} w_{0,2k-1}v_{0,2k-1}(t;q) + \sum_{k=1}^{\infty} w_{0,2k}v_{0,2k}(t)$$
$$+\sum_{m=1}^{\infty} w_{2m-1,0}v_{2m-1,0}(t;q) + \sum_{m,k=1}^{\infty} w_{2m-1,2k-1}v_{2m-1,2k-1}(t;q) + \sum_{m,k=1}^{\infty} w_{2m-1,2k}v_{2m-1,2k}(t;q) \quad (72)$$
$$+\sum_{m=1}^{\infty} w_{2m,0}v_{2m,0}(t;q) + \sum_{m,k=1}^{\infty} w_{2m,2k-1}v_{2m,2k-1}(t;q) + \sum_{m,k=1}^{\infty} w_{2m,2k}v_{2m,2k}(t;q)\Bigg),$$

where

$$q_0(t) = \frac{1}{h(t)}\left(\int_0^1\int_0^1 w(x,y)f(x,y,t)\,dxdy - h''(t)\right),$$

$$w_{l,n} = \int_0^1\int_0^1 \Delta w Z_{l,n}(x,y)\,dxdy, \quad l=\{0,2m-1,2m\},\ n=\{0,2k-1,2k\},\ m,k=1,2,3,\dots,$$

where $v_{0,0}$, $v_{0,2k-1}$, $v_{0,2k}$, $v_{2m-1,0}$, $v_{2m-1,2k-1}$, $v_{2m-1,2k}$, $v_{2m,0}$, $v_{2m,2k-1}$, $v_{2m,2k}$ are defined by the right-hand sides of (30), (32), (34)–(40), respectively. Let's introduce an operator F defines it by the right side (72):

$$F[q](t) = q_0(t) + \frac{1}{h(t)}\Bigg(w_{0,0}v_{0,0}(t;q) + \sum_{k=1}^{\infty} w_{0,2k-1}v_{0,2k-1}(t;q) + \sum_{k=1}^{\infty} w_{0,2k}v_{0,2k}(t;q)$$
$$+\sum_{m=1}^{\infty} w_{2m-1,0}v_{2m-1,0}(t;q) + \sum_{m,k=1}^{\infty} w_{2m-1,2k-1}v_{2m-1,2k-1}(t;q) + \sum_{m,k=1}^{\infty} w_{2m-1,2k}v_{2m-1,2k}(t;q) \quad (73)$$
$$+\sum_{m=1}^{\infty} w_{2m,0}v_{2m,0}(t;q) + \sum_{m,k=1}^{\infty} w_{2m,2k-1}v_{2m,2k-1}(t;q) + \sum_{m,k=1}^{\infty} w_{2m,2k}v_{2m,2k}(t;q)\Bigg).$$

Then the equation (73) can be written in a more convenient form as

$$q(t) = F[q](t), \tag{74}$$

where

$$q_{00} := \max_{t\in[0,T]} |q_0(t)| = \left\| \frac{1}{h(t)}\left(\int_0^1\int_0^1 w(x,y)f(x,y,t)\,dxdy - h''(t)\right)\right\|_{C[0,T]}.$$

We fix a number $\rho > 0$ and consider the ball $B(q_0,\rho) := \{q(t) \in C[0,T] : \|q - q_0\| \leqslant \rho\}$.

First we prove that for sufficiently small $T > 0$ the operator F maps the ball $B(q_0,\rho)$ into itself. Indeed, for any continuous function $q(t)$, the function $F[q](t)$ calculated by the formula (73) will be continuous. At the same time, estimating the norm of differences, we find

$$\big\|F[q](t) - q_0(t)\big\| \leqslant \frac{w_0}{h_0}\Bigg[\left(|\varphi_{0,0,1}| + |\varphi_{0,0,2}|\,T + \frac{\|f_{0,0}\|_{C[0,T]}T^2}{2}\right)\exp\left\{\frac{\|q\|_{C[0,T]}T^2}{2}\right\}$$
$$+\sum_{k=1}^{\infty}\left(|\varphi_{0,2k-1,1}| + \frac{|\varphi_{0,2k-1,2}|}{\sqrt{\gamma_1}} + \frac{\|f_{0,2k-1}\|_{C[0,T]}T}{\gamma_1}\right)\exp\left\{\frac{\|q\|_{C[0,T]}T}{\gamma_1}\right\}$$

$$+\sum_{k=1}^{\infty}\left(|\varphi_{0,2k,1}|+\frac{|\varphi_{0,2k,2}|}{\sqrt{\gamma_1}}+\frac{\|f_{0,2k}\|_{C[0,T]}T}{\gamma_1}+2T\Psi_{0,2k-1}(T)\right)\exp\left\{\frac{\|q\|_{C[0,T]}T}{\gamma_1}\right\}$$

$$+\sum_{m=1}^{\infty}\left(|\varphi_{2m-1,0,1}|+\frac{|\varphi_{2m-1,0,2}|}{\sqrt{\lambda_1}}+\frac{\|f_{2m-1,0}\|_{C[0,T]}T}{\lambda_1}\right)\exp\left\{\frac{\|q\|_{C[0,T]}T}{\lambda_1}\right\}$$

$$+\sum_{m,k=1}^{\infty}\left(|\varphi_{2m-1,2k-1,1}|+\frac{|\varphi_{2m-1,2k-1,2}|}{\sqrt{\mu_{11}}}+\frac{\|f_{2m-1,2k-1}\|_{C[0,T]}T}{\mu_{11}}\right)\exp\left\{\frac{\|q\|_{C[0,T]}T}{\mu_{11}}\right\}$$

$$+\sum_{m,k=1}^{\infty}\left(|\varphi_{2m-1,2k,1}|+\frac{|\varphi_{2m-1,2k,2}|}{\sqrt{\mu_{11}}}+\frac{\|f_{2m-1,2k}\|_{C[0,T]}T}{\mu_{11}}+2T\sqrt{\frac{\gamma_k}{\mu_{11}}}\,\Psi_{2m-1,2k-1}(T)\right)$$

$$\times\exp\left\{\frac{\|q\|_{C[0,T]}T}{\mu_{11}}\right\}+\sum_{m=1}^{\infty}\left(|\varphi_{2m,0,1}|+\frac{|\varphi_{2m,0,2}|}{\sqrt{\lambda_1}}+\frac{\|f_{2m,0}\|_{C[0,T]}T}{\lambda_1}+2T\Psi_{2m-1,0}(T)\right)$$

$$\times\exp\left\{\frac{\|q\|_{C[0,T]}T}{\lambda_1}\right\}+\sum_{m,k=1}^{\infty}\left(|\varphi_{2m,2k-1,1}|+\frac{|\varphi_{2m,2k-1,2}|}{\sqrt{\mu_{11}}}+\frac{\|f_{2m,2k-1}\|_{C[0,T]}T}{\mu_{11}}\right.$$

$$\left.+2T\sqrt{\frac{\lambda_m}{\mu_{11}}}\,\Psi_{2m-1,2k-1}(T)\right)\exp\left\{\frac{\|q\|_{C[0,T]}T}{\mu_{11}}\right\}+\sum_{m,k=1}^{\infty}\left(|\varphi_{2m,2k,1}|+\frac{|\varphi_{2m,2k,2}|}{\sqrt{\mu_{11}}}\right.$$

$$\left.+\frac{\|f_{2m,2k}\|_{C[0,T]}T}{\mu_{11}}+2T\sqrt{\frac{\lambda_m}{\mu_{11}}}\,\Psi_{2m-1,2k}(T)+2T\sqrt{\frac{\gamma_k}{\mu_{11}}}\,\Psi_{2m,2k-1}(T)\right)\exp\left\{\frac{\|q\|_{C[0,T]}T}{\mu_{11}}\right\}\Bigg],$$

where $\omega_0 := \|\omega(x,y)\|_{C^2(\overline{\Omega})}$.

Here we have used the estimate for $v_{0,0}$, $v_{0,2k-1}$, $v_{0,2k}$, $v_{2m-1,0}$, $v_{2m-1,2k-1}$, $v_{2m-1,2k}$, $v_{2m,0}$, $v_{2m,2k-1}$, $v_{2m,2k}$ reduced in (30), (32), (34)–(40). By virtue of the above lemmas, the last series is a convergent series. Note that the function on the right-hand side of this inequality increases monotonically with T_0, and the fact that the function $q(t)$ belongs to the ball $B(q_0,\rho)$ implies the inequality

$$\|q\| \leqslant \|q_0\| + \rho =: R. \tag{75}$$

Therefore, we only strengthen the inequality if we replace $\|q\|$ in it with the expression $\|q_0\|+\rho$. Making these substitutions, we obtain the estimate

$$\big\|F[q](t)-q_0(t)\big\| \leqslant \frac{w_0}{h_0}\Bigg[\left(|\varphi_{0,0,1}|+|\varphi_{0,0,2}|\,T+\frac{\|f_{0,0}\|_{C[0,T]}T^2}{2}\right)\exp\left\{\frac{RT^2}{2}\right\}$$

$$+\sum_{k=1}^{\infty}\left(|\varphi_{0,2k-1,1}|+\frac{|\varphi_{0,2k-1,2}|}{\sqrt{\gamma_1}}+\frac{\|f_{0,2k-1}\|_{C[0,T]}T}{\gamma_1}\right)\exp\left\{\frac{RT}{\gamma_1}\right\}+\sum_{k=1}^{\infty}\left(|\varphi_{0,2k,1}|+\frac{|\varphi_{0,2k,2}|}{\sqrt{\gamma_1}}\right.$$

$$\left.+\frac{\|f_{0,2k}\|_{C[0,T]}T}{\gamma_1}+2T\Psi_{0,2k-1}(T)\right)\exp\left\{\frac{RT}{\gamma_1}\right\}+\sum_{m=1}^{\infty}\left(|\varphi_{2m-1,0,1}|+\frac{|\varphi_{2m-1,0,2}|}{\sqrt{\lambda_1}}\right.$$

$$\left.+\frac{\|f_{2m-1,0}\|_{C[0,T]}T}{\lambda_1}\right)\exp\left\{\frac{RT}{\lambda_1}\right\}+\sum_{m,k=1}^{\infty}\left(|\varphi_{2m-1,2k-1,1}|+\frac{|\varphi_{2m-1,2k-1,2}|}{\sqrt{\mu_{11}}}\right.$$

$$\left.+\frac{\|f_{2m-1,2k-1}\|_{C[0,T]}T}{\mu_{11}}\right)\exp\left\{\frac{RT}{\mu_{11}}\right\}+\sum_{m,k=1}^{\infty}\left(|\varphi_{2m-1,2k,1}|+\frac{|\varphi_{2m-1,2k,2}|}{\sqrt{\mu_{11}}}+\frac{\|f_{2m-1,2k}\|_{C[0,T]}T}{\mu_{11}}\right.$$

$$+2T\sqrt{\frac{\gamma_k}{\mu_{11}}}\,\Psi_{2m-1,2k-1}(T)\Big)\exp\left\{\frac{RT}{\mu_{11}}\right\}+\sum_{m=1}^{\infty}\Big(|\varphi_{2m,0,1}|+\frac{|\varphi_{2m,0,2}|}{\sqrt{\lambda_1}}+\frac{\|f_{2m,0}\|_{C[0,T]}T}{\lambda_1}$$
$$+2T\Psi_{2m-1,0}(T)\Big)\exp\left\{\frac{RT}{\lambda_1}\right\}+\sum_{m,k=1}^{\infty}\Big(|\varphi_{2m,2k-1,1}|+\frac{|\varphi_{2m,2k-1,2}|}{\sqrt{\mu_{11}}}+\frac{\|f_{2m,2k-1}\|_{C[0,T]}T}{\mu_{11}}$$
$$+2T\sqrt{\frac{\lambda_m}{\mu_{11}}}\Psi_{2m-1,2k-1}(T)\Big)\exp\left\{\frac{RT}{\mu_{11}}\right\}+\sum_{m,k=1}^{\infty}\Big(|\varphi_{2m,2k,1}|+\frac{|\varphi_{2m,2k,2}|}{\sqrt{\mu_{11}}}+\frac{\|f_{2m,2k}\|_{C[0,T]}T}{\mu_{11}}$$
$$+2T\sqrt{\frac{\lambda_m}{\mu_{11}}}\,\Psi_{2m-1,2k}(T)+2T\sqrt{\frac{\gamma_k}{\mu_{11}}}\,\Psi_{2m,2k-1}(T)\Big)\exp\left\{\frac{RT}{\mu_{11}}\right\}\Bigg]=:\alpha(T),$$

$\alpha(T)$ is on increasing function.

If we denote by T_1 the positive root of the equation (for T), $\alpha(T)=\rho$. Then $\|F[q](t)-q_0(t)\|\leqslant\rho$ for $T\leqslant T_1$, e. i., those $F[q](t)\in B(q_0,\rho)$.

Now we take any functions $q(t),\widetilde{q}(t)\in B(q_0,\rho)$ and estimate the distance between their images $F[q](t)$ and $F[\widetilde{q}](t)$ in the space $C[0,T]$. The function $v_{m,k}(t)=\widetilde{v}_{m,k}(t)$ corresponding to $\widetilde{q}(t)$ satisfies the integral equations (30), (32), (34)–(40) for $\varphi_{m,k,\mathrm{i}}=\widetilde{\varphi}_{m,k,\mathrm{i}}$ and $f_{m,k}=\widetilde{f}_{m,k}$. Compiling the difference $F[q](t)-F[\widetilde{q}](t)$ using the equations (21)–(29) and then evaluating its norm, we get

$$\big\|F[q](t)-F[\widetilde{q}](t)\big\|\leqslant\frac{w_0}{h_0}\,\Psi_{0,2k-1}(T)\exp\left\{\frac{\|\widetilde{q}\,\|_{C[0,T]}T^2}{2}\right\}\frac{T^2}{2}\,\|\widehat{q}\,\|_{C[0,T]}$$
$$+\frac{w_0}{h_0}\sum_{k=1}^{\infty}\frac{\Psi_{0,2k-1}(T)T}{\gamma_1}\exp\left\{\frac{\|\widetilde{q}\,\|_{C[0,T]}T}{\gamma_1}\right\}\|\widehat{q}\,\|_{C[0,T]}+\frac{w_0}{h_0}\sum_{k=1}^{\infty}\frac{\Psi_{0,2k}(T)}{\gamma_1}$$
$$\times\left(T\exp\left\{\frac{\|\widetilde{q}\,\|_{C[0,T]}T}{\gamma_1}\right\}+2\left[T\exp\left\{\frac{\|\widetilde{q}\,\|_{C[0,T]}T}{\gamma_1}\right\}\right]^2\right)\|\widehat{q}\,\|_{C[0,T]}$$
$$+\frac{w_0}{h_0}\sum_{m=1}^{\infty}\frac{\Psi_{2m-1,0}(T)T}{\lambda_1}\exp\left\{\frac{\|\widetilde{q}\,\|_{C[0,T]}T}{\lambda_1}\right\}\|\widehat{q}\,\|_{C[0,T]}+\frac{w_0}{h_0}\sum_{m,k=1}^{\infty}\frac{\Psi_{2m-1,2k-1}(T)T}{\mu_{11}}$$
$$\times\exp\left\{\frac{\|\widetilde{q}\,\|_{C[0,T]}T}{\mu_{11}}\right\}\|\widehat{q}\,\|_{C[0,T]}+\frac{w_0}{h_0}\sum_{m,k=1}^{\infty}\left[\frac{\Psi_{2m-1,2k}(T)T}{\mu_{11}}\exp\left\{\frac{\|\widetilde{q}\,\|_{C[0,T]}T}{\mu_{11}}\right\}\right.$$
$$\left.+2\sqrt{\frac{\gamma_k}{\mu_{11}^3}}\,\Psi_{2m-1,2k-1}(T)\left(T\exp\left\{\frac{\|\widetilde{q}\,\|_{C[0,T]}T}{\mu_{11}}\right\}\right)^2\right]\|\widehat{q}\,\|_{C[0,T]}+\frac{w_0}{h_0}\sum_{m,k=1}^{\infty}\left[\frac{\Psi_{2m,0}(T)T}{\lambda_1}\right.\tag{76}$$
$$\left.\times\exp\left\{\frac{\|\widetilde{q}\,\|_{C[0,T]}T}{\lambda_1}\right\}+2\,\frac{\Psi_{2m-1,0}(T)}{\lambda_1}\left(T\exp\left\{\frac{\|\widetilde{q}\,\|_{C[0,T]}T}{\lambda_1}\right\}\right)^2\right]\|\widehat{q}\,\|_{C[0,T]}$$
$$+\frac{w_0}{h_0}\sum_{m,k=1}^{\infty}\left[\frac{\Psi_{2m,2k-1}(T)T}{\mu_{11}}\exp\left\{\frac{\|\widetilde{q}\,\|_{C[0,T]}T}{\mu_{11}}\right\}+2\sqrt{\frac{\lambda_m}{\mu_{11}^3}}\,\Psi_{2m-1,2k-1}(T)\right.$$
$$\left.\times\left(T\exp\left\{\frac{\|\widetilde{q}\,\|_{C[0,T]}T}{\mu_{11}}\right\}\right)^2\right]\|\widehat{q}\,\|_{C[0,T]}+\frac{2w_0}{h_0}\sum_{m,k=1}^{\infty}\frac{\sqrt{\lambda_m}+\sqrt{\gamma_k}}{\sqrt{\mu_{11}^3}}\,\Psi_{2m,2k-1}(T)$$
$$\times\left(T\exp\left\{\frac{\|\widetilde{q}\,\|_{C[0,T]}T}{\mu_{11}}\right\}\right)^2\|\widehat{q}\,\|_{C[0,T]}+\frac{w_0}{h_0}\sum_{m,k=1}^{\infty}\frac{\sqrt{\lambda_m\gamma_k}}{\mu_{11}^2}\,\Psi_{2m-1,2k-1}(T)$$

$$\times\left(2T\exp\left\{\frac{\|\widetilde{q}\|_{C[0,T]}T}{\mu_{11}}\right\}\right)^3\|\widehat{q}\|_{C[0,T]}+\frac{w_0}{h_0}\sum_{m,k=1}^{\infty}\frac{\Psi_{2m,2k}(T)T}{\mu_{11}}\exp\left\{\frac{\|\widetilde{q}\|_{C[0,T]}T}{\mu_{11}}\right\}\|\widehat{q}\|_{C[0,T]}.$$

The functions $q(t)$ and $\widetilde{q}(t)$ belong to the ball $B(q_0,\rho)$, so for each of these functions, the inequality (75) is valid. Therefore, replacing $\|q\|$ and $\|\widetilde{q}\|$ in the inequality (76) with $\|q_0\|+\rho$. Thus, we have the right side (76) he

$$\frac{w_0}{h_0}\Psi_{0,2k-1}(T)\exp\left\{\frac{RT^2}{2}\right\}\frac{T^2}{2}\|\widehat{q}\,\|_{C[0,T]}+\frac{w_0}{h_0}\sum_{k=1}^{\infty}\frac{\Psi_{0,2k-1}(T)T}{\gamma_1}\exp\left\{\frac{RT}{\gamma_1}\right\}\|\widehat{q}\,\|_{C[0,T]}$$
$$+\frac{w_0}{h_0}\sum_{k=1}^{\infty}\frac{\Psi_{0,2k}(T)}{\gamma_1}\left(T\exp\left\{\frac{RT}{\gamma_1}\right\}+2\left[T\exp\left\{\frac{RT}{\gamma_1}\right\}\right]^2\right)\|\widehat{q}\,\|_{C[0,T]}$$
$$+\frac{w_0}{h_0}\sum_{m=1}^{\infty}\frac{\Psi_{2m-1,0}(T)T}{\lambda_1}\exp\left\{\frac{RT}{\lambda_1}\right\}\|\widehat{q}\|_{C[0,T]}+\frac{w_0}{h_0}\sum_{m,k=1}^{\infty}\frac{\Psi_{2m-1,2k-1}(T)T}{\mu_{11}}\exp\left\{\frac{RT}{\mu_{11}}\right\}\|\widehat{q}\|_{C[0,T]}$$
$$+\frac{w_0}{h_0}\sum_{m,k=1}^{\infty}\left[\frac{\Psi_{2m-1,2k}(T)T}{\mu_{11}}\exp\left\{\frac{RT}{\mu_{11}}\right\}+2\sqrt{\frac{\gamma_k}{\mu_{11}^3}}\Psi_{2m-1,2k-1}(T)\left(T\exp\left\{\frac{RT}{\mu_{11}}\right\}\right)^2\right]\|\widehat{q}\|_{C[0,T]}$$
$$+\frac{w_0}{h_0}\sum_{m,k=1}^{\infty}\left[\frac{\Psi_{2m,0}(T)T}{\lambda_1}\exp\left\{\frac{RT}{\lambda_1}\right\}+2\frac{\Psi_{2m-1,0}(T)}{\lambda_1}\left(T\exp\left\{\frac{RT}{\lambda_1}\right\}\right)^2\right]\|\widehat{q}\,\|_{C[0,T]}$$
$$+\frac{w_0}{h_0}\sum_{m,k=1}^{\infty}\left[\frac{\Psi_{2m,2k-1}(T)T}{\mu_{11}}\exp\left\{\frac{RT}{\mu_{11}}\right\}+2\sqrt{\frac{\lambda_m}{\mu_{11}^3}}\Psi_{2m-1,2k-1}(T)\left(T\exp\left\{\frac{RT}{\mu_{11}}\right\}\right)^2\right]\|\widehat{q}\|_{C[0,T]}$$
$$+\frac{2w_0}{h_0}\sum_{m,k=1}^{\infty}\frac{\sqrt{\lambda_m}+\sqrt{\gamma_k}}{\sqrt{\mu_{11}^3}}\Psi_{2m,2k-1}(T)\left(T\exp\left\{\frac{RT}{\mu_{11}}\right\}\right)^2\|\widehat{q}\|_{C[0,T]}+\frac{w_0}{h_0}\sum_{m,k=1}^{\infty}\frac{\sqrt{\lambda_m\gamma_k}}{\mu_{11}^2}\Psi_{2m-1,2k-1}(T)$$
$$\left(2T\exp\left\{\frac{RT}{\mu_{11}}\right\}\right)^3\|\widehat{q}\,\|_{C[0,T]}+\frac{w_0}{h_0}\sum_{m,k=1}^{\infty}\frac{\Psi_{2m,2k}(T)T}{\mu_{11}}\exp\left\{\frac{RT}{\mu_{11}}\right\}\|\widehat{q}\,\|_{C[0,T]}=\beta(T)\|\widehat{q}\,\|_{C[0,T]},$$

$\beta(T)$ is an increacing function. We denote T_2 equation $\beta(T)=1$, then for $T\in(0,T_2)$ the operator F shortens the distance between the elements $q(t),\widetilde{q}(t)\in B(q_0,\rho)$. Therefore, if we choose $T^*<\min(T_1,T_2)$, then the operator F is a contraction in the ball $B(q_0,\rho)$. However, in accordance with the Banach theorem [19, pp. 87–97], the operator F has a unique fixed point in the ball $B(q_0,\rho)$, i. e., there exists a unique solution to the equation (74). Theorem 2 is proved. ▷

Let T be a positive fixed number. Consider the set D_{ν_0} of given functions $(\varphi_1,\varphi_2,h,f)$ for which all conditions of Theorem 2 hold and

$$\max\left\{\|\varphi_1\|_{C^4[0,1]},\|\varphi_2\|_{C^3[0,1]},\|h\|_{C^2[0,T]},\|f\|_{C^2(\bar{\Omega})}\right\}\leqslant\nu_0,$$

where ν_0 is a given positive number

Denote by G_{v_1} the set of functions $q(t)$ that for some $T>0$ satisfy the following condition $\|q\|_{C[0,T]}\leqslant\nu_1$, $\nu_1>0$ is a given number.

Theorem 3. *Let* $(\varphi_1,\varphi_2,h,f)\in D_{\nu_0}$, $\left(\widetilde{\varphi}_1,\widetilde{\varphi}_2,\widetilde{h},\widetilde{f}\right)\in D_{\nu_0}$ *and* $q,\widetilde{q}\in G_{\nu_1}$ *Then, for solution to the inverse problem* (1)–(6) *the following stability estimate holds:*

$$\|q-\widetilde{q}\,\|_{C[0,T]}\leqslant r\Big[\|\varphi_1-\widetilde{\varphi}_1\|_{C^4[0,1]}+\|\varphi_2-\widetilde{\varphi}_2\|_{C^3[0,1]}+\|h-\widetilde{h}\|_{C^2[0,T]}+\|f-\widetilde{f}\,\|_{C^2(\bar{\Omega})}\Big],\quad(77)$$

where the constant r depends only on ν_0, ν_1, T.

◁ To prove this theorem, using (72), we write out the equations for $\widetilde{q}(t)$ and form the difference $\widehat{q} = q(t) - \widetilde{q}(t)$. Then after evaluating this expression and using the estimates $v_n(t), \widehat{v}_n(t)$, we obtain the following estimate

$$
\begin{aligned}
\|q-\widetilde{q}\,\|_{C[0,T]} \leqslant \max_{0\leqslant t\leqslant T} \Bigg| \frac{1}{h(t)}\bigg(\int_0^1\!\!\int_0^1 w(x,y)f(x,y,t)\,dxdy - h''(t)\bigg) \\
+\frac{1}{h(t)}\int_0^1\!\!\int_0^1 \Delta w\Bigg(Z_{0,0}(x,y)v_{0,0}(t)+\sum_{k=1}^{\infty}Z_{0,2k-1}(x,y)v_{0,2k-1}(t)+\sum_{k=1}^{\infty}Z_{0,2k}(x,y)v_{0,2k}(t) \\
+\sum_{m=1}^{\infty}Z_{2m-1,0}(x,y)v_{2m-1,0}(t)+\sum_{m,k=1}^{\infty}Z_{2m-1,2k-1}(x,y)v_{2m-1,2k-1}(t) \\
+\sum_{m,k=1}^{\infty}Z_{2m-1,2k}(x,y)v_{2m-1,2k}(t)+\sum_{m=1}^{\infty}Z_{2m,0}(x,y)v_{2m,0}(t) \\
+\sum_{m,k=1}^{\infty}Z_{2m,2k-1}(x,y)v_{2m,2k-1}(t)+\sum_{m,k=1}^{\infty}Z_{2m,2k}(x,y)v_{2m,2k}(t)\Bigg)dxdy \\
-\frac{1}{\widetilde{h}(t)}\bigg(\int_0^1\!\!\int_0^1 w(x,y)\widetilde{f}(x,y,t)\,dxdy-\widetilde{h}''(t)\bigg)-\frac{1}{\widetilde{h}(t)}\int_0^1\!\!\int_0^1\Delta w\Bigg(Z_{0,0}(x,y)\widetilde{v}_{0,0}(t) \\
\times\sum_{k=1}^{\infty}Z_{0,2k-1}(x,y)\widetilde{v}_{0,2k-1}(t)+\sum_{k=1}^{\infty}Z_{0,2k}(x,y)\widetilde{v}_{0,2k}(t)+\sum_{m=1}^{\infty}Z_{2m-1,0}(x,y)\widetilde{v}_{2m-1,0}(t) \\
+\sum_{m,k=1}^{\infty}Z_{2m-1,2k-1}(x,y)\widetilde{v}_{2m-1,2k-1}(t)+\sum_{m,k=1}^{\infty}Z_{2m-1,2k}(x,y)\widetilde{v}_{2m-1,2k}(t) \\
+\sum_{m=1}^{\infty}Z_{2m,0}(x,y)\widetilde{v}_{2m,0}(t)+\sum_{m,k=1}^{\infty}Z_{2m,2k-1}(x,y)\widetilde{v}_{2m,2k-1}(t) \\
+\sum_{m,k=1}^{\infty}Z_{2m,2k}(x,y)\widetilde{v}_{2m,2k}(t)\Bigg)dxdy\Bigg| \leqslant r_0\Big(\|\varphi_1-\widetilde{\varphi}_1\|+\|\varphi_2-\widetilde{\varphi}_2\|+\|f-\widetilde{f}\,\| \\
+\|h''(t)-\widetilde{h}''(t)\|+\|h-\widetilde{h}\,\|\Big)+r_1\int_0^t \sin\sqrt{\mu_{mk}}\,(t-\tau)\|q(\tau)-\widetilde{q}(\tau)\|_{C[0,T]}\,d\tau,\ \ t\in[0,T],
\end{aligned}
\tag{78}
$$

where r_0, r_1 depends only on ν_0, ν_1, T. From (78) with the help of Lemma 1 we obtain the estimate

$$
\begin{aligned}
\|q-\widetilde{q}\,\|_{C[0,T]} \leqslant r_0\Big(\|\varphi_1-\widetilde{\varphi}_1\|_{C^4[0,1]}+\|\varphi_2-\widetilde{\varphi}_2\|_{C^3[0,1]} \\
+\|f-\widetilde{f}\,\|_{C^2(\bar{\Omega})}+\|h-\widetilde{h}\|_{C^2[0,T]}\Big)\exp\left\{\frac{1}{\sqrt{\mu_{11}}}\right\},\quad t\in[0,T].
\end{aligned}
$$

This inequality implies the estimate (77) if we set $r = r_0\exp\{\frac{1}{\sqrt{\mu_{11}}}\}$. Theorem 3 also implies the following assertion about the global uniqueness of the solution of the inverse problem. ▷

Theorem 4. *Let the functions* φ_1, φ_2, h, f *and* $\widetilde{\varphi}_1$, $\widetilde{\varphi}_2$, $\widetilde{h}$, $\widetilde{f}$ *have the same meaning as in Theorem* 2. *Moreover, if* $\varphi_1 = \widetilde{\varphi}_1$, $\varphi_2 = \widetilde{\varphi}_2$, $h = \widetilde{h}$, $f = \widetilde{f}$ *for* $(x, y, t) \in \Omega$, *then* $q(t) = \widetilde{q}(t)$, $t \in [0, T]$.

4. Conclusion

In this paper, we study the solvability of a nonlinear inverse problem for a two-dimensional wave equation with initial boundary conditions. First, we studied the solvability of the initial-boundary problem (1)–(5). The existence, uniqueness, and stability of solutions of the direct problem is proved. We considered the inverse problem of determining the coefficient $q(t)$ of the wave equation in a rectangular domain with an additional integral condition (6). Theorems on local existence and global uniqueness are proved and a stability estimate of the solution is obtained.

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ОБРАТНАЯ КОЭФФИЦИЕНТНАЯ ЗАДАЧА ДЛЯ ДВУМЕРНОГО ВОЛНОВОГО УРАВНЕНИЯ С НАЧАЛЬНЫМИ И НЕЛОКАЛЬНЫМИ ГРАНИЧНЫМИ УСЛОВИЯМИ

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Аннотация. В данной работе рассматриваются прямая и обратная задачи для двумерного волнового уравнения. Прямая задача представляет собой начально-краевую задачу для этого уравнения с нелокальными граничными условиями. В обратной задаче требуется найти переменный во времени коэффициент при младшем члене уравнения. Классическое решение прямой задачи представлено в виде биортогонального ряда по собственным значениям и присоединенным функциям, доказаны единственность и устойчивость этого решения. Для решения обратной задачи получены теоремы существования в локальном, единственности в глобальном и оценка условной устойчивости. Задачи определения правых частей и переменных коэффициентов при младших членах из начально-краевых задач для линейных уравнений в частных производных второго порядка с локальными граничными условиями изучались многими авторами. Поскольку нелинейность является сверхточной, то теоремы об однозначной разрешимости в них доказываются в глобальном смысле. В некоторых работах метод разделения переменных используется для нахождения классического решения прямой задачи в виде биортогонального ряда по собственным функциям и присоединенным функциям. В качестве условия переопределения по отношению к решению прямой задачи используется нелокальное интегральное условие. Прямая задача сводится к эквивалентным интегральным уравнениям метода Фурье. Для установления интегральных неравенств используются обобщенные неравенства типа Гронуолла — Беллмана. Мы получаем априорную оценку решения через неизвестный коэффициент, который нам используется для изучения обратной задачи.

Ключевые слова: волновое уравнение, нелокальные граничные условия, обратная задача, теорема Банаха.

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