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EXISTENCE OF SOLUTIONS FOR A CLASS OF IMPULSIVE BURGERS EQUATION

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Abstract. We study a class of impulsive Burgers equations. A new topological approach is applied to prove the existence of at least one and at least two nonnegative classical solutions. The arguments are based on recent theoretical results. Here we focus our attention on a class of Burgers equations and we investigate it for the existence of classical solutions. The Burgers equation can be used for modeling both traveling and standing nonlinear plane waves. The simplest model equation can describe the second-order nonlinear effects connected with the propagation of high-amplitude (finite-amplitude waves) plane waves and, in addition, the dissipative effects in real fluids. There are several approximate solutions to the Burgers equation. These solutions are always fixed to areas before and after the shock formation. For an area where the shock wave is forming no approximate solution has yet been found. Therefore, it is therefore necessary to solve the Burgers equation numerically in this area.

Keywords: Burgers equation, impulsive Burgers equation, positive solution, fixed point, cone, sum of operators.

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1. Introduction

The Burgers equation is a fundamental partial differential equation in fluid mechanics. It is also a very important model encountered in several areas of applied mathematics such as heat conduction, acoustic waves, gas dynamics and traffic flow. Analytical solutions of the partial differential equations modeling physical phenomena exist only in few of the cases. Therefore the need for the construction of efficient numerical methods for the approximate solution of these models always exists. Many of the analytical solutions to the Burgers equation involve Fourier series. There are several approximate solutions of the Burgers equation (see [1]). These solutions are always fixed to areas before and after the shock formation. For an area where the shock wave is forming no approximate solution has yet been found. It is therefore necessary to solve the Burgers equation numerically in this area (see [2, 3]). Numerical solutions themselves have difficulties with stability and accuracy.

Here, in this paper, we focus our attention on a class of Burgers equations and we will investigate it for existence of classical solutions. More precisely, consider the problem

$$\begin{aligned} u_t + uu_x &= 0, \quad t \in J_0,\ x \in \mathbb{R}, \\ u(0, x) &= u_0(x), \quad x \in \mathbb{R}, \\ u(t_k+, x) &= u(t_k, x) + I_k(u(t_k, x)), \quad x \in \mathbb{R},\ k \in \{1, \ldots, m\}, \end{aligned} \tag{1.1}$$

where

(H1) $T > 0$, $0 = t_0 < t_1 < \ldots < t_m < t_{m+1} = T$, $J = [0, T]$, $J_0 = J \backslash \{t_k\}_{k=1}^m$, $m \in \mathbb{N}$.

(H2) $I_k \in \mathscr{C}([0, T] \times \mathbb{R})$, $|I_k(u)| \leqslant A|u|^{r_k}$, $u \in \mathbb{R}$, $r_k > 0$, $k \in \{1, \ldots, m\}$, A is a positive constant.

(H3) $u_0 \in \mathscr{C}^1(\mathbb{R})$, $|u_0| \leqslant B$ on $\mathbb{R}$, B is a positive constant.

Additional conditions for the constants A and B will be given below. Here

$$u(t_k, x) = \lim_{t \to t_k -} u(t, x), \quad u(t_k+, x) = \lim_{t \to t_k +} u(t, x), x \in \mathbb{R}.$$

Whereas impulsive differential equations are well studied, the literature concerning impulsive partial differential equations does not see to be very rich. To the best of our knowledge, there are no any references devoted on investigations of the impulsive Burgers equation for existence and uniqueness of classical solutions.

The paper is organized as follows. In the next section, we give some preliminary results. In Section 3, we prove existence of at least one solution for the problem (1.1). In Section 4, we prove existence of at least two nonnegative solutions of the problem (1.1). In Section 5, we give an example that illustrates our main results.

2. Preliminary Results

Below, assume that X is a real Banach space. Now, we will recall the definitions of compact and completely continuous mappings in Banach spaces.

Definition 2.1. Let $K : M \subset X \to X$ be a map. We say that K is compact if $K(M)$ is contained in a compact subset of X. The map K is called a completely continuous map if it is continuous and it maps any bounded set into a relatively compact set.

Proposition 2.1 (Leray–Schauder Nonlinear Alternative [4])**.** *Let C be a convex, closed subset of a Banach space E, $0 \in U \subset C$, where U is an open set. Let $f : \overline{U} \to C$ be a continuous, compact map. Then*

(a) *either f has a fixed point in $\overline{U}$;*

(b) *or there exist $x \in \partial U$, and $\lambda \in (0, 1)$, such that $x = \lambda f(x)$.*

To prove our existence result we will use the following fixed point theorem which is a consequence of Proposition 2.1.

Theorem 2.1. *Let E be a Banach space, Y a closed, convex subset of E, U be any open subset of Y with $0 \in U$. Consider two operators T and S, where*

$$Tx = \varepsilon \quad x \in \overline{U},$$

for $\varepsilon > 0$ and $S : \overline{U} \to E$ be such that

(i) *$I - S : \overline{U} \to Y$ continuous, compact and*

(ii) $\{x \in \overline{U} :\ x = \lambda(I - S)x,\ x \in \partial U\} = \varnothing$, *for any* $\lambda \in \left(0, \frac{1}{\varepsilon}\right)$.

Then there exists $x^* \in \overline{U}$ such that

$$Tx^* + Sx^* = x^*.$$

$\lhd$ We have that the operator $\frac{1}{\varepsilon}(I - S) : \overline{U} \to Y$ is continuous and compact. Suppose that there exist $x_0 \in \partial U$ and $\mu_0 \in (0, 1)$, such that

$$x_0 = \mu_0 \frac{1}{\varepsilon}(I - S)x_0,$$

that is

$$x_0 = \lambda_0 (I - S)x_0,$$

where $\lambda_0 = \mu_0 \frac{1}{\varepsilon} \in \left(0, \frac{1}{\varepsilon}\right)$. This contradicts the condition (ii). From Leray–Schauder nonlinear alternative, it follows that there exists $x^* \in \overline{U}$, so that

$$x^* = \frac{1}{\varepsilon}(I - S)x^*,$$

or

$$\varepsilon\, x^* + Sx^* = x^*,$$

or

$$Tx^* + Sx^* = x^*. \rhd$$

Definition 2.2. Let X and Y be real Banach spaces. A map $K : X \to Y$ is called expansive if there exists a constant $h > 1$ for which one has the following inequality

$$\|Kx - Ky\|_Y \geqslant h\|x - y\|_X,$$

for any $x, y \in X$.

Now, we will recall the definition for a cone in a Banach space.

Definition 2.3. A closed, convex set $\mathscr{P}$ in X is said to be cone if

1) $\alpha x \in \mathscr{P}$ for any $\alpha \geqslant 0$ and for any $x \in \mathscr{P}$,

2) $x, -x \in \mathscr{P}$ implies $x = 0$.

Denote $\mathscr{P}^* = \mathscr{P} \backslash \{0\}$. The next result is a fixed point theorem which we will use to prove existence of at least two nonnegative global classical solutions of the IVP (1.1). For its proof, we refer the reader to [5] and [6].

Theorem 2.2. *Let $\mathscr{P}$ be a cone of a Banach space E; Ω a subset of $\mathscr{P}$ and U_1, U_2 and U_3 three open bounded subsets of $\mathscr{P}$, such that $\overline{U}_1 \subset \overline{U}_2 \subset U_3$ and $0 \in U_1$. Assume that $T : \Omega \to \mathscr{P}$ is an expansive mapping, $S : \overline{U}_3 \to E$ is a completely continuous map and $S(\overline{U}_3) \subset (I - T)(\Omega)$. Suppose that $(U_2 \setminus \overline{U}_1) \cap \Omega \neq \varnothing$, $(U_3 \setminus \overline{U}_2) \cap \Omega \neq \varnothing$, and there exists $u_0 \in \mathscr{P}^*$ such that the following conditions hold:*

(i) *$Sx \neq (I - T)(x - \lambda u_0)$, for any $\lambda > 0$ and $x \in \partial U_1 \cap (\Omega + \lambda u_0)$,*

(ii) *there exists $\epsilon \geqslant 0$, such that $Sx \neq (I - T)(\lambda x)$, for any $\lambda \geqslant 1 + \epsilon$, $x \in \partial U_2$ and $\lambda x \in \Omega$,*

(iii) *$Sx \neq (I - T)(x - \lambda u_0)$, for any $\lambda > 0$ and $x \in \partial U_3 \cap (\Omega + \lambda u_0)$.*

Then $T + S$ has at least two non-zero fixed points $x_1, x_2 \in \mathscr{P}$, such that

$$x_1 \in \partial U_2 \cap \Omega \quad \text{and} \quad x_2 \in (\overline{U}_3 \setminus \overline{U}_2) \cap \Omega,$$

or

$$x_1 \in (U_2 \setminus U_1) \cap \Omega \quad \text{and} \quad x_2 \in (\overline{U}_3 \setminus \overline{U}_2) \cap \Omega.$$

Define the spaces $PC(J)$, $PC^1(J)$ and $PC^1(J, \mathscr{C}^1(\mathbb{R}))$ by

$$PC(J) = \Big\{ g: \ g \in \mathscr{C}(J_0), \ \big(\exists\, g(t_j^+), \ g(t_j^-)\big) \text{ and } g(t_j^-) = g(t_j), \ j \in \{1, \dots, k\} \Big\},$$

$$PC^1(J) = \Big\{ g: \ g \in PC(J) \cap \mathscr{C}^1(J_0), \ \big(\exists\, g'(t_j^-), \ g'(t_j^+)\big) \text{ and } g'(t_j^-) = g'(t_j), \ j \in \{1, \dots, k\} \Big\}$$

and

$$PC^1(J, \mathscr{C}^1(\mathbb{R})) = \Big\{ u: \ J \times \mathbb{R} \to \mathbb{R}: \ u(\cdot, x) \in PC^1(J), \ x \in \mathbb{R} \text{ and } u(t, \cdot) \in \mathscr{C}^1(\mathbb{R}), \ t \in J \Big\}. \quad (2.1)$$

Suppose that $X = PC^1(J, \mathscr{C}^1(\mathbb{R}))$ is endowed with the norm

$$\|u\| = \sup \Big\{ \sup_{(t,x) \in [t_j, t_{j+1}] \times \mathbb{R}} |u(t,x)|, \quad \sup_{(t,x) \in [t_j, t_{j+1}] \times \mathbb{R}} |u_x(t,x)|, \quad \sup_{(t,x) \in [t_j, t_{j+1}] \times \mathbb{R}} |u_t(t,x)|, \quad j \in \{1, \dots, k\} \Big\},$$

provided it exists. Note that X is a Banach space. For $u \in X$, define the operator

$$S_1 u(t,x) = u(t,x) - u_0(x) - \sum_{0 < t_k < t} I_k(u(t_k, x)) + \int_0^t u(s,x)\, u_x(s,x)\, ds, \quad (t,x) \in J \times \mathbb{R}.$$

Lemma 2.1. *Suppose that* $(H1)$–$(H3)$ *hold. If* $u \in X$ *satisfies the equation*

$$S_1 u(t,x) = 0, \quad (t,x) \in J \times \mathbb{R}, \quad (2.2)$$

then it is a solution of the problem (1.1).

$\lhd$ Let $u \in X$ be a solution of the equation (2.2). We differentiate the equation (2.2) with respect to t and we find

$$u_t(t,x) + u(t,x) u_x(t,x) = 0,$$

$(t,x) \in J \times \mathbb{R}$, and $u(0,x) = u_0(x)$, $x \in \mathbb{R}$. Next, we put $t = t_j+$ and $t = t_j$, $j \in \{1, \dots, m\}$, in the equation (2.2) and we obtain

$$0 = u(t_j+, x) - u_0(x) - \sum_{0 < t_k < t_j+} I_k(u(t_k, x)) + \int_0^{t_j} u(s,x)\, u_x(s,x)\, ds, \quad j \in \{1, \dots, m\}, \ x \in \mathbb{R},$$

and

$$0 = u(t_j, x) - u_0(x) - \sum_{0 < t_k < t_j} I_k(u(t_k, x)) + \int_0^{t_j} u(s,x)\, u_x(s,x)\, ds, \quad j \in \{1, \dots, m\}, \ x \in \mathbb{R},$$

respectively. Therefore

$$u(t_j+, x) = u(t_j, x) + I_j(u(t_j, x)), \quad j \in \{1, \dots, m\}, \ x \in \mathbb{R}.$$

Consequently u satisfies (1.1). This completes the proof. $\rhd$

Lemma 2.2. *Suppose that* $(H1)$–$(H3)$ *hold. If* $u \in X$, $\|u\| \leqslant B$, *then*

$$|S_1 u(t,x)| \leqslant 2B + A\sum_{k=1}^{m} B^{r_k} + TB^2, \quad (t,x) \in J \times \mathbb{R}.$$

$\lhd$ We have

$$\begin{aligned}|S_1 u(t,x)| &= \left| u(t,x) - u_0(x) - \sum_{0<t_k<t} I_k(u(t_k,x)) + \int_0^t u(s,x)u_x(s,x)\,ds \right| \\ &\leqslant |u(t,x)| + |u_0(x)| + \sum_{0<t_k<t} |I_k(u(t_k,x))| + \int_0^t |u(s,x)u_x(s,x)|\,ds \\ &\leqslant |u(t,x)| + |u_0(x)| + A\sum_{0<t_k<t} |u(t_k,x)|^{r_k} + \int_0^t |u(s,x)u_x(s,x)|\,ds \\ &\leqslant 2B + A\sum_{k=1}^{m} B^{r_k} + TB^2, \quad (t,x) \in J \times \mathbb{R}.\end{aligned}$$

This completes the proof. $\rhd$

(H4) Suppose that $g \in \mathscr{C}(J \times \mathbb{R})$ is a nonnegative function, such that

$$216\left(1+t+t^2+t^3\right)\left(1+|x|+|x|^2+|x|^3+|x|^4+|x|^5+|x|^6\right)\int_0^t \left| \int_0^x g(t_1,x_1)\,dx_1 \right| dt_1 \leqslant D,$$

$(t,x) \in J \times \mathbb{R}$, for some constant $D > 0$. In the last section, we will give an example for a function g and a constant D that satisfy $(H4)$.

For $u \in X$, define the operator

$$S_2 u(t,x) = \int_0^t \int_0^x (t-t_1)^2 (x-x_1)^3 g(t_1,x_1) S_1 u(t_1,x_1)\,dx_1\,dt_1, \quad (t,x) \in J \times \mathbb{R}.$$

Lemma 2.3. *Suppose that* $(H1)$–$(H4)$ *hold. For* $u \in X$, $\|u\| \leqslant B$, *we have*

$$\|S_2 u\| \leqslant D\left(2B + A\sum_{k=1}^{m} B^{r_k} + TB^2\right), \quad (t,x) \in J \times \mathbb{R}.$$

$\lhd$ We have

$$|S_2 u(t,x)| = \left| \int_0^t \int_0^x (t-t_1)^2 (x-x_1)^3 g(t_1,x_1) S_1 u(t_1,x_1)\,dx_1\,dt_1 \right|$$

$$\leqslant \int_0^t \left| \int_0^x (t-t_1)^2 |x-x_1|^3 g(t_1,x_1) |S_1 u(t_1,x_1)|\,dx_1 \right| dt_1 \leqslant \left(2B + A\sum_{k=1}^{m} B^{r_k} + TB^2\right)$$

$$\times \int_0^t \left| \int_0^x (t-t_1)^2 |x-x_1|^3 g(t_1,x_1)(1+t_1)\left(1+|x_1|^2+|x_1|^3\right) dx_1 \right| dt_1$$

$$\leqslant \left(2B + A\sum_{k=1}^{m} B^{r_k} + TB^2\right) t^2(1+t)|x|^3\big(1+|x|^2+|x|^3\big) \int_0^t \left|\int_0^x g(t_1,x_1)\,dx_1\right| dt_1$$

$$\leqslant D\left(2B + A\sum_{k=1}^{m} B^{r_k} + TB^2\right), \quad (t,x)\in J\times\mathbb{R},$$

and

$$\left|\frac{\partial}{\partial t}S_2u(t,x)\right| = \left|2\int_0^t\int_0^x (t-t_1)(x-x_1)^3 g(t_1,x_1)S_1u(t_1,x_1)\,dx_1\,dt_1\right|$$

$$\leqslant 2\int_0^t \left|\int_0^x (t-t_1)|x-x_1|^3 g(t_1,x_1)|S_1u(t_1,x_1)|\,dx_1\right| dt_1$$

$$\leqslant 2\left(2B+A\sum_{k=1}^{m} B^{r_k}+TB^2\right)\int_0^t\left|\int_0^x (t-t_1)|x-x_1|^3 g(t_1,x_1)(1+t_1)\big(1+|x_1|^2+|x_1|^3\big)dx_1\right| dt_1$$

$$\leqslant 72\left(2B + A\sum_{k=1}^{m} B^{r_k} + TB^2\right) t(1+t)|x|^3\big(1+|x|^2+|x|^3\big)\int_0^t\left|\int_0^x g(t_1,x_1)\,dx_1\right| dt_1$$

$$\leqslant D\left(2B + A\sum_{k=1}^{m} B^{r_k} + TB^2\right), \quad (t,x)\in J\times\mathbb{R},$$

and

$$\left|\frac{\partial}{\partial x}S_2u(t,x)\right| = \left|3\int_0^t\int_0^x (t-t_1)^2(x-x_1)^2 g(t_1,x_1)S_1u(t_1,x_1)\,dx_1\,dt_1\right|$$

$$\leqslant 3\int_0^t \left|\int_0^x (t-t_1)^2(x-x_1)^2 g(t_1,x_1)|S_1u(t_1,x_1)|\,dx_1\right| dt_1$$

$$\leqslant 9\left(2B+A\sum_{k=1}^{m} B^{r_k}+TB^2\right)\int_0^t\left|\int_0^x (t-t_1)^2(x-x_1)^2 g(t_1,x_1)(1+t_1)\big(1+|x_1|^2+|x_1|^3\big)\,dx_1\right| dt_1$$

$$\leqslant 108\left(2B + A\sum_{k=1}^{m} B^{r_k} + TB^2\right) t^2(1+t)|x|^2\big(1+|x|^2+|x|^3\big)\int_0^t\left|\int_0^x g(t_1,x_1)\,dx_1\right| dt_1$$

$$\leqslant D\left(2B + A\sum_{k=1}^{m} B^{r_k} + TB^2\right), \quad (t,x)\in J\times\mathbb{R}.$$

Consequently

$$\|S_2u\| \leqslant D\left(2B + A\sum_{k=1}^{m} B^{r_k} + TB^2\right).$$

This completes the proof. $\triangleright$

Lemma 2.4. *Suppose that $(H1)$–$(H4)$ hold. If $u\in X$ satisfies the equation*

$$S_2u(t,x) = C, \quad (t,x)\in J\times\mathbb{R}, \tag{2.3}$$

for some constant C, then u is a solution to the problem (1.1).

◁ We differentiate three times with respect to t the equation (2.3) and we get

$$2\int_0^x (x-x_1)^3 g(t,x_1)S_1u(t,x_1)\,dx_1 = 0, \quad (t,x)\in J\times\mathbb{R},$$

or

$$\int_0^x (x-x_1)^3 g(t,x_1)S_1u(t,x_1) = 0, \quad (t,x)\in J\times\mathbb{R}.$$

Now, we differentiate four times with respect to x the last equation and we find

$$6g(t,x)S_1u(t,x) = 0, \quad (t,x)\in J\times\mathbb{R},$$

or

$$g(t,x)S_1u(t,x) = 0, \quad (t,x)\in J\times\mathbb{R},$$

whereupon

$$S_1u(t,x) = 0, \quad (t,x)\in (0,T]\times(\mathbb{R}\backslash\{0\}).$$

Since $S_1u(\cdot,\cdot)\in\mathscr{C}(J\times\mathbb{R})$, we get

$$0 = \lim_{t\to 0} S_1u(t,x) = S_1u(0,x) = \lim_{x\to 0} S_1u(t,x), \quad (t,x)\in J\times\mathbb{R}.$$

Therefore

$$S_1u(t,x) = 0, \quad (t,x)\in J\times\mathbb{R}.$$

Hence and Lemma 2.1, we conclude that u is a solution to the problem (1.1). This completes the proof. ▷

3. Existence of at Least One Solution

Now, suppose that

(H5) $D\left(2B + A\sum_{k=1}^{m} B^{r_k} + TB^2\right) < B$.

(H6) $\epsilon\left(B + D\left(2B + A\sum_{k=1}^{m} B^{r_k} + TB^2\right)\right) \leqslant B$.

For $u\in X$, define the operators

$$Tu(t,x) = -\epsilon u(t,x),$$

$$Su(t,x) = (1+\epsilon)u(t,x) + \epsilon S_2u(t,x), \quad (t,x)\in J\times\mathbb{R}.$$

By Lemma 2.4, it follows that any fixed point of the operator $T+S$ is a solution to the problem (1.1).

Lemma 3.1. *Suppose that (H1)–(H6) hold. For $u\in X$, we have*

$$\|(I-S)u\| \leqslant B \quad \text{and} \quad \|((1+\epsilon)I - S)u\| < \epsilon B.$$

◁ Applying Lemma 2.3, we get

$$\|(I-S)u\| = \|-\epsilon u - \epsilon S_2u\| \leqslant \epsilon\|u\| + \epsilon\|S_2u\| \leqslant \epsilon\left(B + D\left(2B + A\sum_{k=1}^{m} B^{r_k} + TB^2\right)\right) \leqslant B$$

and

$$\|((1+\epsilon)I-S)u\| = \|\epsilon S_2 u\| = \epsilon\|S_2 u\| \leqslant \epsilon D\left(2B + A\sum_{k=1}^{m} B^{r_k} + TB^2\right) < \epsilon B.$$

This completes the proof. ▷

Our main result in this section is as follows.

Theorem 3.1. *Suppose that* $(H1)$–$(H6)$ *hold. Then the problem* (1.1) *has at least one solution.*

◁ Let $\widetilde{Y}$ denote the set of all equi-continuous families in X with respect to the norm $\|\cdot\|$. Let also, $Y = \overline{\widetilde{Y}}$ be the closure of $\widetilde{Y}$,

$$U = \{u \in Y : \|u\| < B\}.$$

For $u \in \overline{U}$ and $\epsilon > 0$, define the operators

$$T(u)(t,x) = \epsilon u(t,x),$$
$$S(u)(t,x) = u(t,x) - \epsilon u(t,x) - \epsilon S_2(u)(t,x), \quad (t,x) \in J \times \mathbb{R}.$$

For $u \in \overline{U}$, we have

$$\|(I-S)u\| = \|\epsilon u + \epsilon S_2 u\| \leqslant \epsilon\|u\| + \epsilon\|S_2 u\| \leqslant \epsilon B_1 + \epsilon A B_1.$$

Thus, $S : \overline{U} \to X$ is continuous and $(I-S)(\overline{U})$ resides in a compact subset of Y. Now, suppose that there is a $u \in \partial U$, so that

$$u = \lambda(I-S)u,$$

or

$$u = \lambda\epsilon\,(u + S_2 u), \tag{3.1}$$

for some $\lambda \in \left(0, \frac{1}{\epsilon}\right)$. Then, using that $S_2 u(0,x) = 0$, we get

$$u(0,x) = \lambda\epsilon(u(0,x) + S_2 u(0,x)) = \lambda\epsilon u(0,x), \quad x \in \mathbb{R},$$

whereupon $\lambda\epsilon = 1$, which is a contradiction. Consequently

$$\left\{u \in \overline{U} :\ u = \lambda_1(I-S)u,\ u \in \partial U\right\} = \varnothing$$

for any $\lambda_1 \in \left(0, \frac{1}{\epsilon}\right)$. Then, from Theorem 2.1, it follows that the operator $T + S$ has a fixed point $u^* \in Y$. Therefore

$$u^*(t,x) = Tu^*(t,x) + Su^*(t,x) = \epsilon u^*(t,x) + u^*(t,x) - \epsilon u^*(t,x) - \epsilon S_2 u^*(t,x), \quad (t,x) \in J \times \mathbb{R},$$

whereupon

$$S_2 u^*(t,x) = 0, \quad (t,x) \in J \times \mathbb{R}.$$

From here, u^* is a solution to the problem (1.1). From here and from Lemma 2.4, it follows that u is a solution to the IVP (1.1). This completes the proof. ▷

4. Existence of at Least Two Nonnegative Solutions

Suppose:

(H7) Let $m > 0$ be large enough and A, B, $\overset{\approx}{r}$, L, R_1 be positive constants that satisfy the following conditions

$$\overset{\approx}{r} < L < R_1, \quad \epsilon > 0, \quad R_1 > \left(\frac{2}{5m} + 1\right) L,$$

$$D\left(2R_1 + A\sum_{k=1}^{m} R_1^{r_k} + TR_1^2\right) < \frac{L}{5}.$$

For $x \in X$, define the operators

$$T_1u(t,x) = (1+m\epsilon)u(t,x) - \epsilon\frac{L}{10},$$

$$S_3u(t,x) = -\epsilon S_2u(t,x) - m\epsilon u(t,x) - \epsilon\frac{L}{10}, \quad (t,x) \in J\times\mathbb{R}.$$

Our main result in this section is as follows.

Theorem 4.1. *Suppose that* $(H1)$*–*$(H4)$ *and* $(H7)$ *hold. Then the problem* (1.1) *has at least two nonnegative solutions.*

$\lhd$ Define

$$U_1 = \mathscr{P}_{\widetilde{\widetilde{r}}} = \{v \in \mathscr{P}:\ \|v\| < \widetilde{\widetilde{r}}\},$$
$$U_2 = \mathscr{P}_L = \{v \in \mathscr{P}:\ \|v\| < L\},$$
$$U_3 = \mathscr{P}_{R_1} = \{v \in \mathscr{P}:\ \|v\| < R_1\},$$
$$R_2 = R_1 + \frac{D}{m}\left(2R_1 + A\sum_{k=1}^{m} R_1^{r_k} + TR_1^2\right) + \frac{L}{5m},$$
$$\Omega = \overline{\mathscr{P}_{R_2}} = \{v \in \mathscr{P}:\ \|v\| \leqslant R_2\}.$$

1) For $v_1, v_2 \in \Omega$, we have

$$\|T_1v_1 - T_1v_2\| = (1+m\varepsilon)\|v_1 - v_2\|,$$

whereupon $T_1 : \Omega \to E$ is an expansive operator with a constant $1+m\epsilon$.

2) For $v \in \overline{\mathscr{P}}_{R_1}$, we get

$$\|S_3v\| \leqslant \varepsilon\|S_2v\| + m\varepsilon\|v\| + \varepsilon\frac{L}{10} \leqslant \varepsilon\left(D\left(2R_1 + A\sum_{k=1}^{m} R_1^{r_k} + TR_1^2\right) + mR_1 + \frac{L}{10}\right).$$

Therefore $S_3(\overline{\mathscr{P}}_{R_1})$ is uniformly bounded. Since $S_3 : \overline{\mathscr{P}}_{R_1} \to E$ is continuous, we have that $S_3(\overline{\mathscr{P}}_{R_1})$ is equi-continuous. Consequently $S_3 : \overline{\mathscr{P}}_{R_1} \to E$ is a 0-set contraction.

3) Let $v_1 \in \overline{\mathscr{P}}_{R_1}$. Set

$$v_2 = v_1 + \frac{1}{m}S_2v_1 + \frac{L}{5m}.$$

Note that $S_2v_1 + \frac{L}{5} \geqslant 0$ on $J\times\mathbb{R}$. We have $v_2 \geqslant 0$ on $J\times\mathbb{R}$ and

$$\|v_2\| \leqslant \|v_1\| + \frac{1}{m}\|S_2v_1\| + \frac{L}{5m} \leqslant R_1 + \frac{D}{m}\left(2R_1 + A\sum_{k=1}^{m} R_1^{r_k} + TR_1^2\right) + \frac{L}{5m} = R_2.$$

Therefore $v_2 \in \Omega$ and

$$-\varepsilon m v_2 = -\varepsilon m v_1 - \varepsilon S_2v_1 - \varepsilon\frac{L}{10} - \varepsilon\frac{L}{10},$$

or

$$(I - T_1)v_2 = -\varepsilon m v_2 + \varepsilon\frac{L}{10} = S_3v_1.$$

Consequently $S_3(\overline{\mathscr{P}}_{R_1}) \subset (I - T_1)(\Omega)$.

4) Assume that for any $u_0 \in \mathscr{P}^*$ there exist $\lambda \geqslant 0$ and $u \in \partial\mathscr{P}_r \cap (\Omega + \lambda u_0)$ or $u \in \partial\mathscr{P}_{R_1} \cap (\Omega + \lambda u_0)$, such that

$$S_3 u = (I - T_1)(u - \lambda u_0).$$

Then

$$-\epsilon S_2 u - m\epsilon u - \epsilon\frac{L}{10} = -m\epsilon(u - \lambda u_0) + \epsilon\frac{L}{10},$$

or

$$-S_2 u = \lambda m u_0 + \frac{L}{5}.$$

Hence,

$$\|S_2 u\| = \left\|\lambda m u_0 + \frac{L}{5}\right\| > \frac{L}{5}.$$

This is a contradiction.

5) Suppose that for any $\epsilon_1 \geqslant 0$ small enough there exist a $u_1 \in \partial\mathscr{P}_L$ and $\lambda_1 \geqslant 1 + \epsilon_1$, such that $\lambda_1 u_1 \in \overline{\mathscr{P}}_{R_1}$ and

$$S_3 u_1 = (I - T_1)(\lambda_1 u_1). \tag{4.1}$$

In particular, for $\epsilon_1 > \frac{2}{5m}$, we have $u_1 \in \partial\mathscr{P}_L$, $\lambda_1 u_1 \in \overline{\mathscr{P}}_{R_1}$, $\lambda_1 \geqslant 1 + \epsilon_1$ and (4.1) holds. Since $u_1 \in \partial\mathscr{P}_L$ and $\lambda_1 u_1 \in \overline{\mathscr{P}}_{R_1}$, it follows that

$$\left(\frac{2}{5m} + 1\right) L < \lambda_1 L = \lambda_1 \|u_1\| \leqslant R_1.$$

Moreover,

$$-\epsilon S_2 u_1 - m\epsilon u_1 - \epsilon\frac{L}{10} = -\lambda_1 m\epsilon u_1 + \epsilon\frac{L}{10},$$

or

$$S_2 u_1 + \frac{L}{5} = (\lambda_1 - 1) m u_1.$$

From here,

$$2\frac{L}{5} \geqslant \left\|S_2 u_1 + \frac{L}{5}\right\| = (\lambda_1 - 1)m\|u_1\| = (\lambda_1 - 1)mL$$

and

$$\frac{2}{5m} + 1 \geqslant \lambda_1,$$

which is a contradiction.

Therefore all conditions of Theorem 2.2 hold. Hence, the problem (1.1) has at least two solutions u_1 and u_2 so that

$$\|u_1\| = L < \|u_2\| < R_1,$$

or

$$r < \|u_1\| < L < \|u_2\| < R_1.$$

This completes the proof. $\triangleright$

5. An Example

Let $T = 1$, $n = 1$, $m = 3$, $t_1 = \frac{1}{4}$, $t_2 = \frac{1}{3}$, $t_3 = \frac{1}{2}$. Consider the problem

$$u_t + u u_x = 0, \quad t \in \left[0, \frac{1}{4}\right) \cup \left(\frac{1}{4}, \frac{1}{3}\right) \cup \left(\frac{1}{3}, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right], \quad x \in \mathbb{R},$$

$$u(0,x) = \frac{1}{1+x^2}, \quad x \in \mathbb{R},$$
$$u(t_k+,x) = u(t_k,x) + (u(t_k,x))^4, \quad x \in \mathbb{R},\ k \in \{1,2,3\}.$$

Here

$$A = 1, \quad B = 1, \quad C = 1, \quad r = 1.$$

Then

$$2B + A\sum_{k=1}^{m} B^{r_k} + TB^2 = 2 + 3 + 1 = 6.$$

Take

$$D = \epsilon = \frac{1}{10^{50}}, \quad \widetilde{r} = \frac{3}{2}, \quad R_1 = 1, \quad \approx\!\!\!\!\!r = \frac{1}{8}, \quad L = \frac{1}{2}, \quad m = 10^{50}.$$

Then

$$D\left(2B + A\sum_{k=1}^{m} B^{r_k} + TB^2\right) = \frac{6}{10^{50}} < 1 = B$$

and

$$\epsilon\left(B + D\left(2B + A\sum_{k=1}^{m} B^{r_k} + TB^2\right)\right) = \frac{1}{10^{50}}\left(1 + \frac{6}{10^{50}}\right) < 1 = B.$$

Thus, $(H5)$ and $(H6)$ hold. Hence and Theorem 3.1, it follows that the considered problem has at least one solution. Moreover,

$$\approx\!\!\!\!\!r < L < r_1, \quad R_1 = 1 > \left(\frac{2}{5\cdot 10^{50}} + 1\right)\frac{1}{2} = \left(\frac{2}{5m} + 1\right)L,$$

and

$$D\left(2R_1 + A\sum_{k=1}^{m} R_1^{r_k} + TR_1^2\right) = \frac{6}{10^{50}} < \frac{1}{10} = \frac{L}{5}.$$

Consequently $(H7)$ holds. By Theorem 4.1, it follows that the considered problem has at least two nonnegative solutions.

Now, we will construct a function g for arbitrary n. Let

$$h(s) = \log\frac{1 + s^{11}\sqrt{2} + s^{22}}{1 - s^{11}\sqrt{2} + s^{22}}, \quad l(s) = \arctan\frac{s^{11}\sqrt{2}}{1 - s^{22}}, \quad s \in \mathbb{R}.$$

Then

$$h'(s) = \frac{22\sqrt{2}s^{10}(1 - s^{22})}{(1 - s^{11}\sqrt{2} + s^{22})(1 + s^{11}\sqrt{2} + s^{22})}, \quad l'(s) = \frac{11\sqrt{2}s^{10}(1 + s^{20})}{1 + s^{40}}, \quad s \in \mathbb{R}.$$

Therefore

$$-\infty < \lim_{s\to\pm\infty}\left(1 + |s| + \cdots + s^6\right)h(s) < \infty,$$
$$-\infty < \lim_{s\to\pm\infty}\left(1 + |s| + \cdots + s^6\right)l(s) < \infty.$$

Hence, there exists a positive constant C_1 so that

$$\left(1 + |s| + s^2 + \cdots + s^6\right)\left(\frac{1}{44\sqrt{2}}\log\frac{1 + s^{11}\sqrt{2} + s^{22}}{1 - s^{11}\sqrt{2} + s^{22}} + \frac{1}{22\sqrt{2}}\arctan\frac{s^{11}\sqrt{2}}{1 - s^{22}}\right) \leqslant C_1,$$

$s \in \mathbb{R}$. Note that by [7, p. 707, Integral 79], we have

$$\int \frac{dz}{1+z^4} = \frac{1}{4\sqrt{2}} \log \frac{1+z\sqrt{2}+z^2}{1-z\sqrt{2}+z^2} + \frac{1}{2\sqrt{2}} \arctan \frac{z\sqrt{2}}{1-z^2}.$$

Let

$$Q(s) = \frac{s^{10}}{(1+s^2)^4(1+s^{44})(1+s+s^2)^2}, \quad s \in \mathbb{R}.$$

Then there exists a positive constant C_2 so that

$$216\big(1+t+t^2+t^3\big)\left(1+|x|+\cdots+x^6\right)\int\limits_0^t Q(s)\,ds \left|\int\limits_0^x Q(y)\,dy\right| \leqslant C_2, \quad (t,x) \in J \times \mathbb{R}.$$

Take $g(t,x) = \frac{D}{C_2}Q(t)Q(x)$, $(t,x) \in J \times \mathbb{R}$. Hence,

$$216\big(1+t+t^2+t^3\big)\left(1+|x|+\cdots+x^6\right)\int\limits_0^t \left|\int\limits_0^x g(s,y)\,dy\right| ds \leqslant D, \quad (t,x) \in J \times \mathbb{R}.$$

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СУЩЕСТВОВАНИЕ РЕШЕНИЙ ДЛЯ ОДНОГО КЛАССА ИМПУЛЬСНЫХ УРАВНЕНИЙ БЮРГЕРСА

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Аннотация. Мы изучаем класс импульсных уравнений Бюргерса. Для доказательства существования хотя бы одного и хотя бы двух неотрицательных классических решений применяется новый топологический подход. Обоснования опираются на недавние теоретические результаты. Основное внимание уделяется классу уравнений Бюргерса и вопросу существования классических решений. Уравнение Бюргерса можно использовать для моделирования как бегущих, так и стоячих нелинейных плоских волн. Простейшее модельное уравнение способно описать нелинейные эффекты второго порядка, связанные с распространением плоских волн большой амплитуды (волн конечной амплитуды), а также диссипативные эффекты в реальных жидкостях. Существует несколько приближенных решений уравнения Бюргерса. Эти решения всегда фиксируются до и после образования ударной волны. Для области формирования ударной волны приближенное решение пока не найдено. Поэтому в этой области необходимо численное решение уравнения Бюргерса.

Ключевые слова: уравнение Бюргерса, импульсное уравнение Бюргерса, положительное решение, неподвижная точка, конус, сумма операторов.

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