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HARDY TYPE INEQUALITIES IN CLASSICAL AND GRAND LEBESGUE SPACES $L_{p)}$, $0 < p \leqslant 1$, FOR QUASI-MONOTONE FUNCTIONS#

A. Ouardani[1] and A. Senouci[1]

[1] University of Tiaret, Department of Mathematics,
Zaârora, Tiaret 14000, Algeria

E-mail: ouardaniabderrahmane@yahoo.com, kamer295@yahoo.fr

Abstract. In 2020 Rovshan A. Bandaliev et al. proved the boundedness of Hardy operator for monotone functions in grand Lebesgue spaces $L_{p)}(0,1)$, $0 < p \leqslant 1$. In particular, they established similar results for the Hardy operator in classical weighted Lebesgue spaces. Moreover, it is proved that the grand Lebesgue space $L_{p)}(0,1)$ is a quasi-Banach function space. In this work, we are interested in Hardy inequalities applied to quasi-monotonic functions in classical Lebesgue spaces and grand Lebesgue spaces. We establish the boundedness of Hardy operator for quasi-monotone functions in grand Lebesgue spaces $L_{p)},\ w(0,1)$, $0 < p \leqslant 1$. In addition some integral inequalities for the Hardy operator are proved in classical weighted Lebesgue spaces $L_{p,w}(0,1)$, $0 < p < 1$, for quasi-monotone functions. All inequalities are proved with sharp constants. Some results of Rovshan A. Bandaliev et al. are deduced as particular cases. Also other estimates are obtained in classical Lebesgue spaces for Hardy's operator and its dual.

Keywords: inequalities, quasi-monotone functions, Hardy operators, grand Lebesgue spaces, weighted Lebesgue spaces.

AMS Subject Classification: 26D10, 26D15.

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1. Introduction

For $0 < p < \infty$ we denote $L_{p,w}(0,1)$ the set of all Lebesgue measurable functions, such that

$$\|f\|_{L_{p,w}(0,1)} = \|f\|_{p,w} = \left(\int_0^1 |f(x)|^p\, w(x)\, dx\right)^{\frac{1}{p}} < \infty, \tag{1.1}$$

where $w \in L_1^{loc}(0,1)$ and $w(x) > 0$, a. e.

In 1992 T. Iwainiec and C. Sbordone [1] introduced a new type of function spaces $L_{p)}(\Omega)$, $1 < p < \infty$, where Ω is a bounded open set $\Omega \subset \mathbb{R}^n$, called grand Lebesgue spaces. Namely, the grand Lebesgue spaces are defined as the space of the Lebesgue mesurable functions f on Ω such that

$$\|f\|_{p)} = \sup_{0<\varepsilon<p-1} \left(\frac{\varepsilon}{|\Omega|} \int_\Omega |f(x)|^{p-\varepsilon}\, dx\right)^{\frac{1}{p-\varepsilon}} < \infty,$$

where $|\Omega|$ is the Lebesgue measure of Ω.

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These spaces were intensively studied during the last years due to different applications (see [2] and [3]) and continue to attract attention of researchers (see [4–6]).

We state the following definitions,proposition and corollary that are useful in the proofs of main results.

Definition 1 [7]. Let $0 < p \leqslant 1$. We say that function f belongs to the grand Lebesgue space $L_{p)}(0,1)$, if f is non-negative and Lebesgue measurable a. e. on $(0,1)$ for which

$$\|f\|_{L_{p)}(0,1)} = \sup_{0<\varepsilon<\frac{p}{2}} \left(\varepsilon \int_0^1 |f(x)|^{p-\varepsilon}\, dx \right)^{\frac{1}{p-\varepsilon}} < \infty.$$

Definition 2 [7]. Let $0 < p \leqslant 1$. We denote by $\mathscr{A}_p$ the class of measurable functions $f \in L_{p)}(0,1)$, such that

$$\|f\|_{\mathscr{A}_p} = \sup_{0<\varepsilon<\frac{p}{2}} \left(\varepsilon \int_0^1 \left(x^{p-\varepsilon-1} - 1\right) f^{p-\varepsilon}(x)\, dx \right)^{\frac{1}{p-\varepsilon}} < \infty.$$

Remark 1. In [7] was proved that for $0 < p \leqslant 1$, $L_{p)}(0,1)$ is quasi-Banach function space over $(0,1)$. In this case if $w \equiv 1$, (1.1) becomes quasi-norm of usual Lebesgue space $L_p(0,1)$.

The following definition is well-known (see [8]).

Definition 3. We say that a function f is quasimonotone on $]0,\infty[$, if for some real number α, $x^\alpha f(x)$ is a decreasing or an increasing function of x. More precisely, given $\beta \in \mathbb{R}$, we say that $f \in Q_\beta$ if $x^{-\beta} f(x)$ is non-increasing and $f \in Q^\beta$ if $x^{-\beta} f(x)$ is non-decreasing.

The following proposition was proved in [8].

Proposition 1. *Let $-\infty < \beta < +\infty$ and $0 < p \leqslant 1$.*

(a) *Let $f \in Q_\beta$, $0 \leqslant a < b \leqslant \infty$ for $\beta > -1$ and $0 < a < b \leqslant \infty$ for $\beta \leqslant -1$.*

If $\beta \neq -1$, then

$$\left(\int_a^b f(t)\, dt \right)^p \leqslant p\, |\beta + 1|^{1-p} \int_a^b \left(\frac{\left|t^{\beta+1} - a^{\beta+1}\right|}{t^\beta} \right)^{p-1} f^p(t)\, dt. \tag{1.2}$$

If $\beta = -1$, then

$$\left(\int_a^b f(t)\, dt \right)^p \leqslant p \int_a^b \left(t \ln \frac{t}{a} \right)^{p-1} f^p(t)\, dt. \tag{1.3}$$

The inequalities hold in the reversed direction if $1 \leqslant p < \infty$.

(b) *Let $f \in Q^\beta$ and $0 \leqslant a < b \leqslant \infty$ for $\beta < -1$ and $0 \leqslant a < b < \infty$ for $\beta \geqslant -1$.*

If $\beta \neq -1$, then

$$\left(\int_a^b f(t)\, dt \right)^p \leqslant p\, |\beta + 1|^{1-p} \int_a^b \left(\frac{\left|t^{\beta+1} - b^{\beta+1}\right|}{t^\beta} \right)^{p-1} f^p(t)\, dt. \tag{1.4}$$

If $\beta = -1$, then

$$\left(\int_a^b f(t)\, dt \right)^p \leqslant p \int_a^b \left(t \ln \frac{b}{t} \right)^{p-1} f^p(t)\, dt. \tag{1.5}$$

The inequalities hold in the reversed direction, if $1 \leqslant p < \infty$.

(c) *The constants in these inequalities are the best possible in all cases.*

If in (1.2), (1.4), (1.3) and (1.5) we set $a = 0$, $b = 1$, $a = 0$, $b = x$, $a = x$, $b = 1$ and $a = x$, $b = 1$ respectively, then we get the following corollary.

Corollary 1. *Let $0 < p \leqslant 1$.*

(a) *If $\beta > -1$, $f \in Q_\beta$, then*

$$\left(\int_0^1 f(t)\,dt\right)^p \leqslant p\,|\beta+1|^{1-p}\int_0^1 t^{p-1} f^p(t)\,dt. \tag{1.6}$$

(b) *If $\beta > -1$, $f \in Q^\beta$, then*

$$\left(\int_0^x f(t)\,dt\right)^p \leqslant p\,|\beta+1|^{1-p}\int_0^x \left(t^{-\beta}\left|t^{\beta+1}-x^{\beta+1}\right|\right)^{p-1} f^p(t)\,dt. \tag{1.7}$$

(c) *If $f \in Q_{-1}$, then*

$$\left(\int_x^1 f(t)\,dt\right)^p \leqslant p\int_x^1 \left(t\ln\frac{t}{a}\right)^{p-1} f^p(t)\,dt. \tag{1.8}$$

(d) *If $f \in Q^{-1}$, then*

$$\left(\int_x^1 f(t)\,dt\right)^p \leqslant p\int_x^1 \left(t\ln\frac{b}{t}\right)^{p-1} f^p(t)\,dt. \tag{1.9}$$

The constants in these inequalities are the best possible.

2. Main Results

Throughout the paper, we will assume that the functions are non-negative and Lebesgue measurable on $(0,1)$. We consider the Hardy operators

$$(H_1 f)(x) = \frac{1}{x}\int_0^x f(t)\,dt, \quad (H_2 f)(x) = \frac{1}{x}\int_x^1 f(t)\,dt.$$

Theorem 1. *Let $0 < p < 1$, $\beta > -1$, $w(x) = x^{p-1} - 1$, $0 < x < 1$ and $f \in Q_\beta$. Then the inequality*

$$\|H_1 f\|_{L_p(0,1)} \leqslant \left[(\beta+1)^{1-p}\,\frac{p}{1-p}\right]^{\frac{1}{p}} \|f\|_{L_{p,w}(0,1)} \tag{2.1}$$

holds, where $\left[(\beta+1)^{1-p}\,\frac{p}{1-p}\right]^{\frac{1}{p}}$ is the sharp constant (the best possible).

◁ By applying Corollary 1 (a), we obtain

$$\|H_1 f\|_{L_p(0,1)} = \left(\int_0^1 (H_1 f)^p(x)\,dx\right)^{\frac{1}{p}} = \left(\int_0^1 \frac{1}{x^p}\left(\int_0^x f(t)\,dt\right)^p dx\right)^{\frac{1}{p}}$$

$$= \left[\int_0^1 \frac{1}{x^p}\left(\int_0^1 f(t)\chi_{(0,x)}(t)\,dt\right)^p dx\right]^{\frac{1}{p}} \leqslant p^{\frac{1}{p}}(\beta+1)^{\frac{1-p}{p}}\left[\int_0^1 \frac{1}{x^p}\left(\int_0^1 f^p(t)\chi_{(0,x)}(t)t^{p-1}\,dt\right) dx\right]^{\frac{1}{p}}.$$

Now, by the Fubini theorem, we get

$$\|H_1 f\|_{L_p(0,1)} \leqslant p^{\frac{1}{p}} (\beta+1)^{\frac{1-p}{p}} \left[\int_0^1 f^p(t) t^{p-1} \left(\int_t^1 \frac{dx}{x^p} \right) dt \right]^{\frac{1}{p}}$$

$$= \left(\frac{p}{1-p} \right)^{\frac{1}{p}} (\beta+1)^{\frac{1-p}{p}} \left(\int_0^1 f^p(t) t^{p-1} \left(1 - t^{1-p}\right) dt \right)^{\frac{1}{p}}$$

$$= \left(\frac{p}{1-p} \right)^{\frac{1}{p}} (\beta+1)^{\frac{1-p}{p}} \left(\int_0^1 \left(t^{p-1} - 1\right) f^p(t)\, dt \right)^{\frac{1}{p}},$$

thus

$$\|H_1 f\|_{L_p(0,1)} \leqslant \left[\left(\frac{p}{1-p} \right) (\beta+1)^{1-p} \right]^{\frac{1}{p}} \|f\|_{L_{p,w}(0,1)}.$$

Let $f(x) = (\beta+1)^{\frac{p-1}{p}} x^\beta$. Indeed,

$$\|H_1 f\|_{L_p(0,1)} = \left(\int_0^1 \frac{1}{x^p} \left(\int_0^x f(t)\, dt \right)^p dx \right)^{\frac{1}{p}} = \left[\int_0^1 \frac{1}{x^p} \left(\int_0^x (\beta+1)^{\frac{p-1}{p}} t^\beta\, dt \right)^p dx \right]^{\frac{1}{p}}$$

$$= (\beta+1)^{\frac{p-1}{p}} \left(\int_0^1 x^{-p} \frac{x^{(\beta+1)p}}{(\beta+1)^p}\, dx \right)^{\frac{1}{p}} = (\beta+1)^{-\frac{1}{p}} \left(\int_0^1 x^{\beta p}\, dx \right)^{\frac{1}{p}} = (\beta+1)^{-\frac{1}{p}} \left(\frac{1}{\beta p+1} \right)^{\frac{1}{p}}$$

and

$$\|f\|_{L_{p,w}(0,1)} = \left(\int_0^1 \left(t^{p-1} - 1\right) f^p(t)\, dt \right)^{\frac{1}{p}} = \left(\int_0^1 \left(t^{p-1} - 1\right) (\beta+1)^{p-1} t^{\beta p}\, dt \right)^{\frac{1}{p}}$$

$$= (\beta+1)^{\frac{p-1}{p}} \left(\int_0^1 \left(t^{\beta p+p-1} - t^{\beta p} \right) dt \right)^{\frac{1}{p}} = (\beta+1)^{\frac{p-1}{p}} \left(\frac{1}{\beta p+p} - \frac{1}{\beta p+1} \right)^{\frac{1}{p}}$$

$$= (\beta+1)^{\frac{p-1}{p}} \left(\frac{1-p}{p} \right)^{\frac{1}{p}} \left(\frac{1}{\beta+1} \right)^{\frac{1}{p}} \left(\frac{1}{\beta p+1} \right)^{\frac{1}{p}},$$

so

$$\left[\left(\frac{p}{1-p} \right) (\beta+1)^{1-p} \right]^{\frac{1}{p}} \|f\|_{L_{p,w}(0,1)}$$

$$= \left[\left(\frac{p}{1-p} \right) (\beta+1)^{1-p} \right]^{\frac{1}{p}} (\beta+1)^{\frac{p-1}{p}} \left(\frac{1-p}{p} \right)^{\frac{1}{p}} \left(\frac{1}{\beta+1} \right)^{\frac{1}{p}} \left(\frac{1}{\beta p+1} \right)^{\frac{1}{p}}$$

$$= (\beta+1)^{-\frac{1}{p}} \left(\frac{1}{\beta p+1} \right)^{\frac{1}{p}}.$$

The proof is complete. $\triangleright$

REMARK 2. If $\beta = 0$ in (2.1), we have Theorem 1 of [7].

Theorem 2. *Let $0 < p < 1$, $\beta > -1$, $w(x) = \int_x^1 \frac{(1-t^{\beta+1})^{p-1}}{t^{\beta(p-1)+1}}\,dt$ and $f \in Q^\beta$. Then the inequality*

$$\|H_1 f\|_{L_p(0,1)} \leqslant \left(p\,(\beta+1)^{1-p}\right)^{\frac{1}{p}} \|f\|_{L_{p,w}(0,1)}\,, \tag{2.2}$$

holds, where $\left(p(\beta+1)^{1-p}\right)^{\frac{1}{p}}$ is the sharp constant.

$\lhd$ By using Corollary 1 (b) and the Fubini theorem, we have

$$\|H_1 f\|_{L_p(0,1)} = \left(\int_0^1 \frac{1}{x^p}\left(\int_0^x f(t)\,dt\right)^p dx\right)^{\frac{1}{p}}$$

$$\leqslant p^{\frac{1}{p}}(\beta+1)^{\frac{1-p}{p}}\left[\int_0^1 \frac{1}{x^p}\left(\int_0^x \left[t^{-\beta}\left|x^{\beta+1}-t^{\beta+1}\right|\right]^{p-1} f^p(t)\,dt\right)dx\right]^{\frac{1}{p}}$$

$$= p^{\frac{1}{p}}(\beta+1)^{\frac{1-p}{p}}\left[\int_0^1 x^{-p}\left(\int_0^1 \left[t^{-\beta}\left(x^{\beta+1}-t^{\beta+1}\right)\right]^{p-1} f^p(t)\chi_{(0,x)}(t)\,dt\right)dx\right]^{\frac{1}{p}}$$

$$= p^{\frac{1}{p}}(\beta+1)^{\frac{1-p}{p}}\left[\int_0^1 f^p(t)t^{-\beta(p-1)}\left(\int_t^1 x^{-p}\left(x^{\beta+1}-t^{\beta+1}\right)^{p-1}dx\right)dt\right]^{\frac{1}{p}}.$$

Let $x = \frac{t}{y}$, thus

$$\|H_1 f\|_{L_p(0,1)} \leqslant p^{\frac{1}{p}}(\beta+1)^{\frac{1-p}{p}}\left[\int_0^1 f^p(t)t^{-\beta(p-1)}\left(\int_t^1 t^{-p}y^p\left(t^{\beta+1}y^{-\beta-1}-t^{\beta+1}\right)^{p-1} t y^{-2}\,dy\right)dt\right]^{\frac{1}{p}}$$

$$= p^{\frac{1}{p}}(\beta+1)^{\frac{1-p}{p}}\left[\int_0^1 f^p(t)t^{-\beta(p-1)}t^{-p+1}t^{(\beta+1)(p-1)}\left(\int_t^1 y^{p-2}\left(\frac{1-y^{\beta+1}}{y^{\beta+1}}\right)^{p-1}dy\right)dt\right]^{\frac{1}{p}}$$

$$= p^{\frac{1}{p}}(\beta+1)^{\frac{1-p}{p}}\left[\int_0^1 f^p(t)\left(\int_t^1 \frac{\left(1-y^{\beta+1}\right)^{p-1}}{y^{\beta(p-1)+1}}\,dy\right)dt\right]^{\frac{1}{p}} = p^{\frac{1}{p}}(\beta+1)^{\frac{1-p}{p}}\|f\|_{L_{p,w}(0,1)}.$$

Let $f(x) = (\beta+1)^{\frac{p-1}{p}}x^\beta$. Indeed,

$$\|H_1 f\|_{L_p(0,1)} = \left[\int_0^1 \frac{1}{x^p}\left(\int_0^x (\beta+1)^{\frac{p-1}{p}}t^\beta\,dt\right)^p dx\right]^{\frac{1}{p}} = (\beta+1)^{-\frac{1}{p}}\left(\frac{1}{\beta p+1}\right)^{\frac{1}{p}}$$

and

$$\|f\|_{L_{p,w}(0,1)} = \left(\int_0^1 (\beta+1)^{p-1} x^{\beta p} \left(\int_x^1 \frac{\left(1-t^{\beta+1}\right)^{p-1}}{t^{\beta(p-1)+1}}\, dt \right) dx \right)^{\frac{1}{p}}$$

$$= (\beta+1)^{\frac{p-1}{p}} \left(\int_0^1 \frac{\left(1-t^{\beta+1}\right)^{p-1}}{t^{\beta(p-1)+1}} \left(\int_0^t x^{\beta p}\, dx \right) dt \right)^{\frac{1}{p}}$$

$$= (\beta+1)^{\frac{p-1}{p}} (\beta p+1)^{-\frac{1}{p}} \left(\int_0^1 t^{\beta} \left(1-t^{\beta+1}\right)^{p-1} dt \right)^{\frac{1}{p}}$$

$$= (\beta+1)^{\frac{p-1}{p}} (\beta p+1)^{-\frac{1}{p}} (\beta+1)^{-\frac{1}{p}} \left(\int_0^1 (\beta+1) t^{\beta} \left(1-t^{\beta+1}\right)^{p-1} dt \right)^{\frac{1}{p}}$$

$$= \left(p(\beta+1)^{2-p}(\beta p+1)\right)^{-\frac{1}{p}},$$

so

$$\left(p(\beta+1)^{1-p}\right)^{\frac{1}{p}} \|f\|_{L_{p,w}(0,1)} = \left(p(\beta+1)^{1-p}\right)^{\frac{1}{p}} \left(p(\beta+1)^{2-p}(\beta p+1)\right)^{-\frac{1}{p}}$$

$$= (\beta+1)^{-\frac{1}{p}} (\beta p+1)^{-\frac{1}{p}}.$$

The proof is complete. $\rhd$

REMARK 3. If $\beta = 0$ in (2.2), we get Theorem 2 of [7].

Theorem 3. *Let $0 < p < 1$, $0 < a < b \leqslant \infty$, $w(t) = \left(\ln\left(\frac{t}{a}\right)\right)^{p-1}\left(1 - t^{p-1}a^{1-p}\right)$, $0 < t < 1$, and $f \in Q_{-1}$. Then the inequalty*

$$\|H_2 f\|_{L_p(0,1)} \leqslant \left(\frac{p}{1-p}\right)^{\frac{1}{p}} \|f\|_{L_{p,w}(0,1)}, \tag{2.3}$$

holds, where $\left(\frac{p}{1-p}\right)^{\frac{1}{p}}$ is the sharp constant.

$\lhd$ By applying Corollary 1 (c) and the Fubini theorem, we obtain

$$\|H_2 f\|_{L_p(0,1)} = \left(\int_a^1 (H_2 f)^p(x)\, dx \right)^{\frac{1}{p}} = \left(\int_a^1 \frac{1}{x^p} \left(\int_x^1 f(t)\, dt \right)^p dx \right)^{\frac{1}{p}}$$

$$\leqslant \left[\int_a^1 \frac{1}{x^p} \left(p \int_x^1 \left(t \ln\left(\frac{t}{a}\right) \right)^{p-1} f^p(t)\, dt \right) dx \right]^{\frac{1}{p}}$$

$$= p^{\frac{1}{p}} \left[\int_a^1 \left(t \ln\left(\frac{t}{a}\right) \right)^{p-1} f^p(t) \left(\int_a^t x^{-p}\, dx \right) dt \right]^{\frac{1}{p}}$$

$$= \left(\frac{p}{1-p}\right)^{\frac{1}{p}} \left(\int_a^1 f^p(t) \left(\ln\left(\frac{t}{a}\right) \right)^{p-1} \left(1 - t^{p-1}a^{1-p}\right) dt \right)^{\frac{1}{p}},$$

thus

$$\|H_2 f\|_{L_p(0,1)} \leqslant \left(\frac{p}{1-p}\right)^{\frac{1}{p}} \|f\|_{L_{p,w}(0,1)}\,.$$

Finally, we obtain the required inequality.

We suppose that there exists $C > 0$, such that $C \leqslant \left(\frac{p}{1-p}\right)^{\frac{1}{p}}$, thus $C_1 = C^p(1-p) \leqslant p$, then one can conclude that exists C_1, $C_1 \leqslant p$, which contradicts the fact that p is the smallest possible in (1.8). $\rhd$

Theorem 4. *Let $0 < p < 1$, $0 \leqslant a < b < \infty$, $w_1(t) = \left(\ln\left(\frac{b}{t}\right)\right)^{p-1}$, $0 < t < 1$, and $f \in Q^{-1}$. Then the inequalty*

$$\|H_2 f\|_{L_p(0,1)} \leqslant \left(\frac{p}{1-p}\right)^{\frac{1}{p}} \|f\|_{L_{p,w}(0,1)}\,, \tag{2.4}$$

holds, where $\left(\frac{p}{1-p}\right)^{\frac{1}{p}}$ is the sharp constant.

$\lhd$ The proof follows in view of Corollary 1 (d) and the rest is similar to that of Theorem 3. $\rhd$

Now we lead with the Hardy operator in the grand Lebesgue spaces.

By Definition 1, we have $0 < p \leqslant 1$ and $0 < \varepsilon < \frac{p}{2}$, thus $0 < p-\varepsilon < 1$, then one can apply Corollary 1 by replacing p by $p-\varepsilon$. Consequently we get the following statements.

Corollary 2. *Let $0 < p < 1$, $0 < \varepsilon < \frac{p}{2}$.*

(a) *If $\beta > -1$, $f \in Q_\beta$ and $0 \leqslant a < b \leqslant \infty$, then*

$$\left(\int_0^b f(y)\,dy\right)^{p-\varepsilon} \leqslant (p-\varepsilon)\,(\beta+1)^{1-p+\varepsilon} \left(\int_0^b y^{p-\varepsilon-1} f^{p-\varepsilon}(y)\,dy\right). \tag{2.5}$$

(b) *If $\beta > -1$, $f \in Q^\beta$, then*

$$\left(\int_0^x f(y)\,dy\right)^{p-\varepsilon} \leqslant (p-\varepsilon)\,(\beta+1)^{1-p+\varepsilon} \int_0^x \left[y^{-\beta}\left(x^{\beta+1} - y^{\beta+1}\right)\right]^{p-\varepsilon-1} f^{p-\varepsilon}(y)\,dy. \tag{2.6}$$

The constants in these inequalities are the best possible.

Theorem 5. *Let $0 < p < 1$, $0 < \varepsilon < \frac{p}{2}$, $f \in \mathscr{A}_p$ and $f \in Q_\beta$, $\beta \geqslant 0$. Then*

$$\|H_1 f\|_{L_{p)}(0,1)} \leqslant C\,\|f\|_{\mathscr{A}_p}\,. \tag{2.7}$$

If $C > 0$ is the sharp constant in (2.7), then

$$\left(\frac{p}{2-p}\right)^{\frac{2}{p}} \leqslant C \leqslant (\beta+1)^{\frac{2}{p}-1}\left(\frac{p}{1-p}\right)^{\frac{1}{p}}. \tag{2.8}$$

$$\lhd\ \|H_1 f\|_{L_{p)}(0,1)} = \sup_{0<\varepsilon<\frac{p}{2}} \left(\varepsilon \int_0^1 |H_1 f(t)|^{p-\varepsilon}\,dt\right)^{\frac{1}{p-\varepsilon}} = \sup_{0<\varepsilon<\frac{p}{2}} \left[\varepsilon \int_0^1 \frac{1}{x^{p-\varepsilon}} \left(\int_0^x f(t)\,dt\right)^{p-\varepsilon} dx\right]^{\frac{1}{p-\varepsilon}}.$$

By using Corollary 2 (a) with $b = x$, we obtain

$$\|H_1 f\|_{L_{p)}(0,1)} \leqslant \sup_{0<\varepsilon<\frac{p}{2}} (p-\varepsilon)^{\frac{1}{p-\varepsilon}} (\beta+1)^{\frac{1-p+\varepsilon}{p-\varepsilon}} \left[\varepsilon \int_0^1 x^{\varepsilon-p} \left(\int_0^x t^{p-\varepsilon-1} f^{p-\varepsilon}(t)\chi_{(0,1)}(t)\,dt\right) dx\right]^{\frac{1}{p-\varepsilon}}$$

$$= \sup_{0<\varepsilon<\frac{p}{2}} (p-\varepsilon)^{\frac{1}{p-\varepsilon}} (\beta+1)^{\frac{1-p+\varepsilon}{p-\varepsilon}} \left[\varepsilon \int_0^1 t^{p-\varepsilon-1} f^{p-\varepsilon}(t) \left(\int_t^1 x^{\varepsilon-p} dx\right) dt\right]^{\frac{1}{p-\varepsilon}}$$

$$= \sup_{0<\varepsilon<\frac{p}{2}} (p-\varepsilon)^{\frac{1}{p-\varepsilon}} (\beta+1)^{\frac{1-p+\varepsilon}{p-\varepsilon}} \left(\frac{1}{1+\varepsilon-p}\right)^{\frac{1}{p-\varepsilon}} \left[\varepsilon \int_0^1 \left(1-t^{1+\varepsilon-p}\right) t^{p-\varepsilon-1} f^{p-\varepsilon}(t)\,dt\right]^{\frac{1}{p-\varepsilon}}$$

$$= \sup_{0<\varepsilon<\frac{p}{2}} \left(\frac{p-\varepsilon}{1+\varepsilon-p}\right)^{\frac{1}{p-\varepsilon}} (\beta+1)^{\frac{1-p+\varepsilon}{p-\varepsilon}} \left(\varepsilon \int_0^1 \left(t^{p-\varepsilon-1}-1\right) f^{p-\varepsilon}(t)\,dt\right)^{\frac{1}{p-\varepsilon}}$$

$$\leqslant \sup_{0<\varepsilon<\frac{p}{2}} (\beta+1)^{\frac{1-p+\varepsilon}{p-\varepsilon}} \sup_{0<\varepsilon<\frac{p}{2}} \left(\frac{p-\varepsilon}{1+\varepsilon-p}\right)^{\frac{1}{p-\varepsilon}} \sup_{0<\varepsilon<\frac{p}{2}} \left(\varepsilon \int_0^1 \left(t^{p-\varepsilon-1}-1\right) f^{p-\varepsilon}(t)\,dt\right)^{\frac{1}{p-\varepsilon}}.$$

Let $0 < \varepsilon < \frac{p}{2}$, thus $1-p+\varepsilon < 1-p+\frac{p}{2} = 1-\frac{p}{2}$, therefore $\frac{1-p+\varepsilon}{p-\varepsilon} < \frac{2}{p} - 1$.

Since the function $\varepsilon \mapsto l(\varepsilon) = \left(\frac{p-\varepsilon}{1+\varepsilon-p}\right)^{\frac{1}{p-\varepsilon}}$ is decreasing on interval $(0, \frac{p}{2})$, then we obtain

$$\|H_1 f\|_{L_{p)}(0,1)} \leqslant (\beta+1)^{\frac{2}{p}-1} \left(\frac{p}{1-p}\right)^{\frac{1}{p}} \|f\|_{\mathscr{A}_p}.$$

One can deduce that

$$C \leqslant (\beta+1)^{\frac{2}{p}-1} \left(\frac{p}{1-p}\right)^{\frac{1}{p}}.$$

On the other hand, let us proved the left hand side of (2.8).

Let $f(x) = \beta + 1$, thus

$$\|f\|_{\mathscr{A}_p} = \|(\beta+1)\|_{\mathscr{A}_p} = \sup_{0<\varepsilon<\frac{p}{2}} \left(\varepsilon \int_0^1 \left(x^{p-\varepsilon-1}-1\right)(\beta+1)^{p-\varepsilon}\,dx\right)^{\frac{1}{p-\varepsilon}}$$

$$= \sup_{0<\varepsilon<\frac{p}{2}} \varepsilon^{\frac{1}{p-\varepsilon}} (\beta+1) \left(\frac{1}{p-\varepsilon} - 1\right)^{\frac{1}{p-\varepsilon}} = \sup_{0<\varepsilon<\frac{p}{2}} \varepsilon^{\frac{1}{p-\varepsilon}} (\beta+1) \left(\frac{\varepsilon-p+1}{p-\varepsilon}\right)^{\frac{1}{p-\varepsilon}}$$

$$\leqslant \sup_{0<\varepsilon<\frac{p}{2}} \varepsilon^{\frac{1}{p-\varepsilon}} \sup_{0<\varepsilon<\frac{p}{2}} \left(\frac{\varepsilon-p+1}{p-\varepsilon}\right)^{\frac{1}{p-\varepsilon}} (\beta+1) = \sup_{0<\varepsilon<\frac{p}{2}} \varepsilon^{\frac{1}{p-\varepsilon}} \left(\frac{2-p}{p}\right)^{\frac{2}{p}} (\beta+1),$$

and

$$\|H_1 f\|_{L_{p)}(0,1)} = \|H_1(\beta+1)\|_{p)} = \sup_{0<\varepsilon<\frac{p}{2}} \left[\varepsilon \int_0^1 \frac{1}{x^{p-\varepsilon}} \left(\int_0^x (\beta+1) dt\right)^{p-\varepsilon} dx\right]^{\frac{1}{p-\varepsilon}}$$

$$= \sup_{0<\varepsilon<\frac{p}{2}} \varepsilon^{\frac{1}{p-\varepsilon}} (\beta+1),$$

then by (2.7)

$$C \geqslant \frac{\|H_1 f\|_{L_{p)}(0,1)}}{\|f\|_{\mathscr{A}_{p)}}} = \frac{\sup\limits_{0<\varepsilon<\frac{p}{2}} \varepsilon^{\frac{1}{p-\varepsilon}} (\beta+1)}{\sup\limits_{0<\varepsilon<\frac{p}{2}} \varepsilon^{\frac{1}{p-\varepsilon}} \left(\frac{2-p}{p}\right)^{\frac{2}{p}} (\beta+1)} = \left(\frac{p}{2-p}\right)^{\frac{2}{p}}.$$

The proof is complete. $\rhd$

Remark 4. If $\beta = 0$ in (2.8), we have Theorem 3 of [7].

Theorem 6. *Let $0 < p < 1$, $0 < \varepsilon < \frac{p}{2}$, $w(t) = \int_t^1 \frac{(1-y^{\beta+1})^{p-\varepsilon-1}}{y^{\beta(p-\varepsilon-1)+1}}\,dy$, $0 < y < 1$, and $f \in Q^\beta$, $\beta \geqslant 0$. Then the inequality*

$$\|H_1 f\|_{L_{p)}(0,1)} \leqslant C\,\|f\|_{L_{p),w}(0,1)}, \tag{2.9}$$

holds, where

$$\|f\|_{L_{p),w}(0,1)} = \sup_{0<\varepsilon<\frac{p}{2}} \left(\varepsilon \int_0^1 \left(\int_t^1 \frac{(1-y^{\beta+1})^{p-\varepsilon-1}}{y^{\beta(p-\varepsilon-1)+1}} dy\right) f^{p-\varepsilon}(t)\,dt\right)^{\frac{1}{p-\varepsilon}}.$$

If $C > 0$ is the sharp constant in (2.9), then

$$\left(\frac{p}{2}\right)^{\frac{2}{p}} \leqslant C \leqslant \left((\beta+1)^{1-\frac{p}{2}} p\right)^{\frac{1}{p}}. \tag{2.10}$$

$$\lhd\ \|H_1 f\|_{L_{p)}(0,1)} = \sup_{0<\varepsilon<\frac{p}{2}} \left(\varepsilon \int_0^1 |H_1 f(t)|^{p-\varepsilon}\,dt\right)^{\frac{1}{p-\varepsilon}} = \sup_{0<\varepsilon<\frac{p}{2}} \left[\varepsilon \int_0^1 \frac{1}{x^{p-\varepsilon}} \left(\int_0^x f(t)\,dt\right)^{p-\varepsilon} dx\right]^{\frac{1}{p-\varepsilon}}.$$

According to Corollary 2 (b) and the Fubini theorem, we have

$$\|H_1 f\|_{L_{p)}(0,1)} \leqslant \sup_{0<\varepsilon<\frac{p}{2}} \left[\varepsilon \int_0^1 \frac{1}{x^{p-\varepsilon}} \left((p-\varepsilon)(\beta+1)^{1-p+\varepsilon} \int_0^x \left(t^{-\beta}\left(x^{\beta+1} - t^{\beta+1}\right)\right)^{p-\varepsilon-1}\right.\right.$$

$$\left.\left. \times f^{p-\varepsilon}(t)\chi_{(0,1)}(t)\,dt\right) dx\right]^{\frac{1}{p-\varepsilon}} = \sup_{0<\varepsilon<\frac{p}{2}} \left((p-\varepsilon)(\beta+1)^{1-p+\varepsilon}\right)^{\frac{1}{p-\varepsilon}}$$

$$\times \left[\varepsilon \int_0^1 f^{p-\varepsilon}(t) t^{-\beta(p-\varepsilon-1)} \left(\int_t^1 \left(x^{\beta+1} - t^{\beta+1}\right)^{p-\varepsilon-1} x^{\varepsilon-p}\,dx\right) dt\right]^{\frac{1}{p-\varepsilon}}.$$

Let $x = \frac{t}{y}$, then

$$\left[\varepsilon \int_0^1 f^{p-\varepsilon}(t) t^{-\beta(p-\varepsilon-1)} \left(\int_t^1 \left(x^{\beta+1} - t^{\beta+1}\right)^{p-\varepsilon-1} x^{\varepsilon-p}\,dx\right) dt\right]^{\frac{1}{p-\varepsilon}}$$

$$= \left[\varepsilon \int_0^1 f^{p-\varepsilon}(t) t^{-\beta(p-\varepsilon-1)} \left(\int_1^t \left(\left(\frac{t}{y}\right)^{\beta+1} - t^{\beta+1}\right)^{p-\varepsilon-1} \left(\frac{t}{y}\right)^{\varepsilon-p} \left(-\frac{t}{y^2}\right) dy\right) dt\right]^{\frac{1}{p-\varepsilon}}$$

$$= \left[\varepsilon\int_0^1 f^{p-\varepsilon}(t)t^{-\beta(p-\varepsilon-1)}t^{(\beta+1)(p-\varepsilon-1)}t^{\varepsilon-p+1}\left(\int_t^1\left(\left(\frac{1}{y}\right)^{\beta+1}-1\right)^{p-\varepsilon-1}\left(\frac{1}{y}\right)^{\varepsilon-p}\left(\frac{1}{y^2}\right)dy\right)dt\right]^{\frac{1}{p-\varepsilon}}$$

$$= \left[\varepsilon\int_0^1 f^{p-\varepsilon}(t)\left(\int_t^1\frac{\left(1-y^{\beta+1}\right)^{p-\varepsilon-1}}{y^{(\beta+1)(p-\varepsilon-1)+\varepsilon-p+2}}\,dy\right)dt\right]^{\frac{1}{p-\varepsilon}}$$

$$= \left[\varepsilon\int_0^1 f^{p-\varepsilon}(t)\left(\int_t^1\frac{\left(1-y^{\beta+1}\right)^{p-\varepsilon-1}}{y^{\beta(p-\varepsilon-1)+1}}\,dy\right)dt\right]^{\frac{1}{p-\varepsilon}} \leqslant \|f\|_{L_{p),w}(0,1)},$$

so

$$\|H_1 f\|_{L_{p)}(0,1)} \leqslant \sup_{0<\varepsilon<\frac{p}{2}}\left((p-\varepsilon)(\beta+1)^{1-p+\varepsilon}\right)^{\frac{1}{p-\varepsilon}}\left(\varepsilon\int_0^1 f^{p-\varepsilon}(t)\left(\int_t^1\frac{\left(1-y^{\beta+1}\right)^{p-\varepsilon-1}}{y^{\beta(p-\varepsilon-1)+1}}\,dy\right)dt\right)^{\frac{1}{p-\varepsilon}}$$

$$\leqslant p^{\frac{1}{p}}\left((\beta+1)^{1-\frac{p}{2}}\right)^{\frac{1}{p}}\sup_{0<\varepsilon<\frac{p}{2}}\left(\varepsilon\int_0^1 f^{p-\varepsilon}(t)\left(\int_t^1\frac{\left(1-y^{\beta+1}\right)^{p-\varepsilon-1}}{y^{\beta(p-\varepsilon-1)+1}}dy\right)dt\right)^{\frac{1}{p-\varepsilon}}.$$

In the right hand side of (2.9), it's obvious that $C\leqslant p^{\frac{1}{p}}\left((\beta+1)^{1-\frac{p}{2}}\right)^{\frac{1}{p}}$. Since for all $y\in]0,1[$; $\left(1-y^{\beta+1}\right)^{1+\varepsilon-p}\geqslant(1-y)^{1+\varepsilon-p}$, therefore

$$\int_t^1\frac{1}{\left(1-y^{\beta+1}\right)^{1+\varepsilon-p}}\,dy=\lim_{\alpha\to 1}\int_t^\alpha\frac{1}{\left(1-y^{\beta+1}\right)^{1+\varepsilon-p}}\,dy\leqslant\lim_{\alpha\to 1}\int_t^\alpha\frac{1}{(1-y)^{1+\varepsilon-p}}\,dy.$$

By putting $f(t)=\beta+1$ and taking in account $\frac{1}{p-\epsilon+1}<\frac{1}{1+\frac{p}{2}}<1$, we get

$$\sup_{0<\varepsilon<\frac{p}{2}}\left(\varepsilon\int_0^1(\beta+1)^{p-\varepsilon}\left(\int_t^1\left(1-y^{\beta+1}\right)^{p-\varepsilon-1}dy\right)dt\right)^{\frac{1}{p-\varepsilon}}$$

$$=(\beta+1)\sup_{0<\varepsilon<\frac{p}{2}}\varepsilon^{\frac{1}{p-\varepsilon}}\left(\int_0^1\left(\lim_{\alpha\to1}\int_t^\alpha\left(1-y^{\beta+1}\right)^{p-\varepsilon-1}dy\right)dt\right)^{\frac{1}{p-\varepsilon}}$$

$$\leqslant(\beta+1)\sup_{0<\varepsilon<\frac{p}{2}}\varepsilon^{\frac{1}{p-\varepsilon}}\left(\int_0^1\left(\lim_{\alpha\to1}\int_t^\alpha(1-y)^{p-\varepsilon-1}dy\right)dt\right)^{\frac{1}{p-\varepsilon}}$$

$$=(\beta+1)\sup_{0<\varepsilon<\frac{p}{2}}\varepsilon^{\frac{1}{p-\varepsilon}}\left(\frac{1}{p-\varepsilon}\right)^{\frac{1}{p-\varepsilon}}\left(\int_0^1(1-t)^{p-\varepsilon}dt\right)^{\frac{1}{p-\varepsilon}}$$

$$\leqslant(\beta+1)\sup_{0<\varepsilon<\frac{p}{2}}\varepsilon^{\frac{1}{p-\varepsilon}}\sup_{0<\varepsilon<\frac{p}{2}}\left(\frac{1}{p-\varepsilon}\right)^{\frac{1}{p-\varepsilon}}\left(\frac{1}{p-\varepsilon+1}\right)^{\frac{1}{p-\varepsilon}}\leqslant(\beta+1)\sup_{0<\varepsilon<\frac{p}{2}}\varepsilon^{\frac{1}{p-\varepsilon}}\left(\frac{2}{p}\right)^{\frac{2}{p}}.$$

On the other hand

$$\|H_1 f\|_{L_{p)}(0,1)}=\|H_1(\beta+1)\|_{L_{p)}(0,1)}=(\beta+1)\sup_{0<\varepsilon<\frac{p}{2}}\varepsilon^{\frac{1}{p-\varepsilon}},$$

by (2.9), we conclude that

$$(\beta+1)\sup_{0<\varepsilon<\frac{p}{2}}\varepsilon^{\frac{1}{p-\varepsilon}}\leqslant C(\beta+1)\sup_{0<\varepsilon<\frac{p}{2}}\varepsilon^{\frac{1}{p-\varepsilon}}\left(\frac{2}{p}\right)^{\frac{2}{p}},$$

thus $C\geqslant\left(\frac{p}{2}\right)^{\frac{2}{p}}$. The proof is complete. ▷

Remark 5. If $\beta=0$ in (2.10), we have Theorem 4 of [7].

A similar results hold for $p=1$.

Corollary 3. *Let $f\in L_{1)}(0,1)$, $f\in Q^{\beta}$, $\beta\geqslant 0$. Then there exists a constant $C>0$, such that*

$$\|H_1 f\|_{L_{1)}(0,1)}\leqslant C\sup_{0<\varepsilon<\frac{p}{2}}\left(\varepsilon\int_0^1\left(\int_t^1\frac{(1-y^{\beta+1})^{-\varepsilon}}{y^{-\beta\varepsilon+1}}\,dy\right)f^{1-\varepsilon}(t)\,dt\right)^{\frac{1}{1-\varepsilon}}. \qquad (2.11)$$

If C is the best constant in (2.11), then

$$\frac{1}{4}\leqslant C\leqslant(\beta+1)^{\frac{1}{2}}. \qquad (2.12)$$

Remark 6. If $\beta=0$ in (2.12), we find Theorem 6 of [7].

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Abderrahmane Ouardani
University of Tiaret, Department of Mathematics,
Zaârora, Tiaret 14000, Algeria,
Associate Professor
E-mail: ouardaniabderrahmane@yahoo.com

Senouci Abdelkader
University of Tiaret, Department of Mathematics,
Zaârora, Tiaret 14000, Algeria,
Professor
E-mail: `kamer295@yahoo.fr`
https://orcid.org/0000-0002-3620-7455

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НЕРАВЕНСТВА ТИПА ХАРДИ В КЛАССИЧЕСКОМ И ГРАНД-ПРОСТРАНСТВАХ ЛЕБЕГА $L_{p)}$, $0 < p \leqslant 1$, ДЛЯ КВАЗИМОНОТОННЫХ ФУНКЦИЙ

Уардани А.[1], Сенучи А.[1]

[1] Университет Тиарет, Алжир, 14000, Тиарет, Заарора

E-mail: `ouardaniabderrahmane@yahoo.com`, `kamer295@yahoo.fr`

Аннотация. В 2020 г. Ровшан А. Бандалиев и др. доказал ограниченность оператора Харди для монотонных функций в гранд-пространствах Лебега $L_{p)}(0,1)$, $0 < p \leqslant 1$. В частности, они установили аналогичные результаты для оператора Харди в классических весовых лебеговых пространствах. Более того, доказано, что гранд-пространство Лебега $L_{p)}(0,1)$ является квазибанаховым функциональным пространством. В данной работе нас интересуют неравенства Харди, применяемые к квазимонотонным функциям в классических пространствах Лебега и гранд-пространствах Лебега. Установлена ограниченность оператора Харди для квазимонотонных функций в гранд-пространствах Лебега $L_{p)}$, $w(0,1)$, $0 < p \leqslant 1$. Кроме того, некоторые интегральные неравенства для оператора Харди доказаны в классических весовых пространствах Лебега $L_{p,w}(0,1)$, $0 < p < 1$, для квазимонотонных функций. Все неравенства доказываются с точными константами. Некоторые результаты Ровшана А. Бандалиева и др. выводятся как частные случаи. Получены и другие оценки в классических пространствах Лебега для оператора Харди и двойственного к нему оператора.

Ключевые слова: неравенства, квазимонотонные функции, операторы Харди, гранд-пространства Лебега, весовые пространства Лебега.

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