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ON THE RATE OF CONVERGENCE OF ERGODIC AVERAGES FOR FUNCTIONS OF GORDIN SPACE#

I. V. Podvigin[1]

[1] Sobolev Institute of Mathematics of the Siberian Branch of the RAS,
4 Ac. Koptug Ave., Novosibirsk 630090, Russia
E-mail: `ipodvigin@math.nsc.ru`

To Georgiĭ Georgievich Magaril–Il'yaev
in occasion of his 80th birthday

Abstract. For an automorphisms with non-zero Kolmogorov-Sinai entropy, a new class of L_2-functions called the Gordin space is considered. This space is the linear span of Gordin classes constructed by some automorphism-invariant filtration of σ-algebras $\mathfrak{F}_n$. A function from the Gordin class is an orthogonal projection with respect to the operator $I - E(\cdot|\mathfrak{F}_n)$ of some $\mathfrak{F}_m$-measurable function. After Gordin's work on the use of the martingale method to prove the central limit theorem, this construction was developed in the works of Volný. In this review article we consider this construction in ergodic theory. It is shown that the rate of convergence of ergodic averages in the L_2 norm for functions from the Gordin space is simply calculated and is $\mathscr{O}(\frac{1}{\sqrt{n}})$. It is also shown that the Gordin space is a dense set of the first Baire category in $L_2(\Omega, \mathfrak{F}, \mu) \ominus L_2(\Omega, \Pi(T, \mathfrak{F}), \mu)$, where $\Pi(T, \mathfrak{F})$ is the Pinsker σ-algebra.

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1. Introduction

In the famous work of Gordin [1] the martingale approximation method was first used to prove the central limit theorem for stationary sequences. Subsequently, this approach was developed both for the central limit theorem (see, for example, [2–4]), and in other problems (see, for example, [5] on the convergence of series and [6–8] on large deviations); see also review [9]. The key idea is to consider the filtration of σ-algebras associated with a measure-preserving transformation. We will use this construction in the theory of convergence rates in ergodic theorems.

Let $(\Omega, \mathfrak{F}, \mu)$ be a standard probability space, and let $T : \Omega \to \Omega$ be a measurable invertible measure-preserving transformation (automorphism). Let $\mathfrak{F}_0$ be a σ-subalgebra of σ-algebra $\mathfrak{F}$, such that $T^{-1}\mathfrak{F}_0 \subseteq \mathfrak{F}_0$. Thus, a filtration of σ-algebras $\mathfrak{F}_n := T^n\mathfrak{F}_0$, $n \in \mathbb{Z}$ arises, i. e.,

$$\mathfrak{F}_n \subseteq \mathfrak{F}_{n+1}, \quad n \in \mathbb{Z}.$$

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Let us denote by E_n the conditional expectation operator with respect to the σ-algebra $\mathfrak{F}_n$, acting in $L_2(\Omega, \mathfrak{F}, \mu)$, i. e., $E_n f = \mathbb{E}(f|\mathfrak{F}_n)$. The operator E_n orthogonally projects $L_2(\Omega, \mathfrak{F}, \mu)$ onto $L_2(\Omega, \mathfrak{F}_n, \mu)$. We will also use the symbol T to denote the Koopman operator acting in $L_2(\Omega, \mathfrak{F}, \mu)$ according to the rule $Tf = f \circ T$. Let us also denote ergodic averages as $A_n^T f$, i. e.,

$$A_n^T f = \frac{1}{n} \sum_{k=0}^{n-1} f \circ T^k, \quad f \in L_2(\Omega, \mathfrak{F}, \mu).$$

Von Neumann's ergodic theorem states that $A_n^T f$ norm-converges to $\mathbb{E}(f|\mathfrak{J})$ for $n \to \infty$, where $\mathfrak{J}$ is the σ-algebra of T-invariant sets, i. e., such $A \in \mathfrak{F}$ that $T^{-1}A = A$.

It is well known that functions f from the class of coboundaries (or cohomologous to zero), i. e., $f \in (I - T)L_2(\Omega, \mathfrak{F}, \mu)$, are characterized by the rate of convergence $\frac{1}{n}$ in von Neumann's ergodic theorem. Namely for such and only such functions the asymptotic relation $\|A_n^T f\|_2 = \mathscr{O}\left(\frac{1}{n}\right)$ for $n \to \infty$ holds [10]. It is worth noting that for non-unitary operators T the asymptotics $\|A_n^T f\|_2 = o\left(\frac{1}{\sqrt{n}}\right)$ as $n \to \infty$ with some additional condition will also imply zero cohomology [11].

The class of coboundaries is believed to be the only simple construction for the abstract transformation T, where an estimate of the rate of convergence in von Neumann's ergodic theorem is easily obtained. There are also more complex classes of functions for which it is possible to obtain estimates of the rates of convergence of ergodic averagess, for example, the class of fractional coboundaries [12].

Our goal in this somewhat survey article is to consider in some sense new class of functions for which it is quite simple to obtain an estimate of the rate of convergence in von Neumann's ergodic theorem. We also prove that when the transformation is a K-automorphism, this class is dense in the space of of L_2 functions with zero integral.

We say that a function f belongs to the Gordin class $\mathfrak{G}(T, \mathfrak{F}_0)$ generated by the σ-algebra $\mathfrak{F}_0$, if

$$f \in (I - E_n)L_2(\Omega, \mathfrak{F}_m, \mu), \quad \text{i. e.,} \quad f \in L_2(\Omega, \mathfrak{F}_m, \mu) \ominus L_2(\Omega, \mathfrak{F}_n, \mu)$$

for some $m, n \in \mathbb{Z}$, $m > n$. Thus,

$$\mathfrak{G}(T, \mathfrak{F}_0) = \bigcup_{m>n} H_{n,m}, \quad H_{n,m} = (I - E_n)L_2(\Omega, \mathfrak{F}_m, \mu).$$

The Gordin space $\mathfrak{G}(T)$ is the linear span of all the Gordin classes $\mathfrak{G}(T, \mathfrak{F}_0)$, i. e.,

$$\mathfrak{G}(T) = \operatorname{span}\{\mathfrak{G}(T, \mathfrak{F}_0)\}.$$

Let us mention several results where functions from the Gordin space are found.

Remark 1. The proof of Kolmogorov's theorem that K-automorphisms have infinite Lebesgue spectral type (see, for example, [13, Theorem 5.13]) involves functions from the Gordin class of the form $\chi_A - \mathbb{E}(\chi_A|\mathfrak{F}_{-1})$, $A \in \mathfrak{F}_0$.

Remark 2. It is shown in [14, Theorem 6.1] that functions of the form $g = \chi_A - \mathbb{E}(\chi_A|\mathfrak{F}_{-n})$, $A \in \mathfrak{F}_m$, $m, n > 0$, are Wiener–Wintner functions of power type 1/4 in $L_2(\Omega, \mathfrak{F}, \mu)$. This means that

$$\left\| \sup_{\varepsilon} \left| \frac{1}{N} \sum_{k=0}^{N-1} g \circ T^k \cdot e^{2\pi i k \varepsilon} \right| \right\|_2 = \mathscr{O}\left(\frac{1}{\sqrt[4]{N}} \right).$$

Now let us present the main result.

Theorem 1. *For any function f from the Gordin space $\mathfrak{G}(T)$ the following estimate is true:*

$$\left\|A_N^T f\right\|_2 = \mathscr{O}\left(\frac{1}{\sqrt{N}}\right) \quad \text{as} \quad N \to \infty.$$

2. The Proof of Theorem 1

Let us use the following properties of the conditional expectation. Let $\mathfrak{A}$ be a σ-algebra, then

$$\mathbb{E}(\mathbb{E}(f|\mathfrak{A})) = \mathbb{E}f, \tag{1}$$

$$\mathbb{E}(f \cdot g|\mathfrak{A}) = g \cdot \mathbb{E}(f|\mathfrak{A}), \quad g \text{ — } \mathfrak{A}\text{-measurable}, \tag{2}$$

$$\mathbb{E}(f \circ T|\mathfrak{A}) = \mathbb{E}(f|T\mathfrak{A}) \circ T, \tag{3}$$

where $\mathbb{E}f = \int f d\mu$.

$\lhd$ The set of functions f with $\left\|A_N^T f\right\|_2 = \mathscr{O}\left(\frac{1}{\sqrt{N}}\right)$ is clearly linear. To prove Theorem 1, it is enough to consider the function f from some Gordin class $\mathfrak{G}(T, \mathfrak{F}_0)$. Then there are $n, m \in \mathbb{Z}$ with $m > n$ and $g \in L_2(\Omega, \mathfrak{F}_m, \mu)$, such that $f = g - E_n g$.

To estimate the norm of ergodic averages, we use the following well-known formula (see, for example, [15, § 1]):

$$\left\|A_N^T f\right\|_2^2 = \frac{1}{N^2} \sum_{|k|<N} \big(N - |k|\big)\big(f \circ T^k, f\big)_{L_2} = \frac{\|f\|_2^2}{N} + \frac{2}{N^2} \sum_{k=1}^{N-1} (N-k) \operatorname{Re}\big(f \circ T^k, f\big)_{L_2}.$$

When calculating scalar products, we will use the properties (1), (2) and (3). Taking in account that $g \circ T^k$ will be $\mathfrak{F}_{m-k}$-measurable, and $\mathfrak{F}_{m-k} \subset \mathfrak{F}_n$ for $k \geqslant m - n$, for such k we obtain

$$\begin{aligned}
\big(f \circ T^k, f\big)_{L_2} &= \big(g \circ T^k - E_n g \circ T^k, g - E_n g\big)_{L_2} \\
&= \mathbb{E}\big(g \circ T^k \cdot \bar{g}\big) - \mathbb{E}\big(g \circ T^k \cdot E_n \bar{g}\big) + \mathbb{E}\big(E_n g \circ T^k \cdot E_n \bar{g}\big) - \mathbb{E}\big(E_n g \circ T^k \cdot \bar{g}\big) \\
&= \mathbb{E}\big(g \circ T^k \cdot \bar{g}\big) - \mathbb{E}\big(E_n\big(g \circ T^k \cdot \bar{g}\big)\big) + \mathbb{E}\big(E_{n-k}\big(g \circ T^k\big) \cdot E_n \bar{g}\big) - \mathbb{E}\big(E_{n-k}\big(g \circ T^k\big) \cdot \bar{g}\big) \\
&= \mathbb{E}\big(g \circ T^k \cdot \bar{g}\big) - \mathbb{E}\big(g \circ T^k \cdot \bar{g}\big) + \mathbb{E}\big(E_n\big(E_{n-k}(g \circ T^k\big) \cdot \bar{g}\big)\big) - \mathbb{E}\big(E_{n-k}\big(g \circ T^k\big) \cdot \bar{g}\big) \\
&= 0 + \mathbb{E}(E_{n-k}\big(g \circ T^k\big) \cdot \bar{g}\big) - \mathbb{E}(E_{n-k}\big(g \circ T^k\big) \cdot \bar{g}\big) = 0.
\end{aligned}$$

Thus, for all $N \geqslant m - n > 0$

$$\begin{aligned}
\left\|A_N^T f\right\|_2^2 &= \frac{\|f\|_2^2}{N} + \frac{2}{N^2} \sum_{k=1}^{m-n-1} (N-k) \operatorname{Re}\big(f \circ T^k, f\big)_{L_2} \leqslant \frac{\|f\|_2^2}{N} + \frac{2\|f\|_2^2}{N^2} \sum_{k=1}^{m-n-1} (N-k) \\
&= \frac{\|f\|_2^2}{N} + \frac{2\|f\|_2^2}{N} (m-n-1)\Big(1 - \frac{m-n}{2N}\Big) \leqslant \frac{2\|f\|^2(m-n)}{N}.
\end{aligned}$$

The proof of Theorem 1 is complete. $\rhd$

Let us present several corollaries.

Corollary 1. *For any function f from the Gordin class, the spectral measure σ_f satisfies the estimate $\sigma_f((-\delta, \delta]) = \mathscr{O}(\delta)$ as $\delta \to 0$.*

$\lhd$ This follows from the well-known Kachurovskii criterion (see, for example, [15, Theorem 3]), as well as from the fact that the spectral measure σ_f has a continuously differentiable density (see, [15, Theorem 7]), since most of the correlation coefficients $(f \circ T^k, f)_{L_2}$, $k \in \mathbb{Z}$, vanish. $\triangleright$

Corollary 2. *For uniform convergence on the space $H_{n,m}$ there is the estimate*

$$\|A_N^T\|_{H_{n,m} \to L_2(\Omega, \mathfrak{F}, \mu)} \leqslant \sqrt{\frac{2(m-n)}{N}}$$

for all $N \geqslant m - n$.

In connection with Corollary 2, we note a recent paper [16], in which subspaces with power-law uniform convergence in von Neumann's discrete-time ergodic theorem were studied.

Corollary 3. *For any function f from Gordin space the following asymptotic relation holds a. e.:*

$$A_N^T f = o\left(\frac{\ln N (\ln \ln N)^\beta}{\sqrt{N}}\right) \quad \text{as} \quad N \to \infty$$

for any $\beta > 1/2$.

$\triangleleft$ This follows from Theorem 4.5 in [17]. $\triangleright$

Corollary 4. *If f is in the Gordin space $\mathfrak{G}(T)$, then $f \in (I-T)^\alpha L_2(\Omega, \mathfrak{F}, \mu)$ for every $0 < \alpha < 1/2$. For $\alpha = 1/2$ a similar statement is not true.*

$\triangleleft$ The first statement follows from Theorem 1 and [12, Theorem 2.17]. For $\alpha = 1/2$ we will use the following criterion [12, Theorem 2.11 and Corollary 2.12]:

$$f \in (I-T)^{1/2} L_2(\Omega, \mathfrak{F}, \mu) \iff \sup_{N \geqslant 1} \left\| \sum_{j=1}^{N} \frac{T^j f}{\sqrt{j}} \right\|_2 < \infty.$$

For any nonzero $f \in H_{n,m}$ with $m - n = 1$, using the fact that $(T^k f, f) = 0$ for $k \neq 0$ we have

$$\sup_{N \geqslant 1} \left\| \sum_{j=1}^{N} \frac{T^j f}{\sqrt{j}} \right\|_2^2 = \sup_{N \geqslant 1} \sum_{j=1}^{N} \sum_{i=1}^{N} \frac{(T^j f, T^i f)_{L_2}}{\sqrt{ij}} = \|f\|_2^2 \sup_{N \geqslant 1} \sum_{k=1}^{N} \frac{1}{k} = \infty. \quad \triangleright$$

3. Additional Properties of Gordin Space

Let us now discuss the question for which automorphisms the Gordin class exists, i. e., it does not degenerate into a zero function.

Proposition 1. *The Gordin space $\mathfrak{G}(T) = \{0\}$ if and only if T has zero Kolmogorov–Sinai entropy $h(T)$.*

$\triangleleft$ It is clear that Gordin classes consist only of zero function if and only if all σ-algebras $\mathfrak{F}_n$ in the filtration coincide. This is equivalent to $\mathfrak{F}_0 = T^{-1}\mathfrak{F}_0$, i. e., the σ-algebra $\mathfrak{F}_0$ is invariant under T, or is a factor. Thus, we need to find a condition on the automorphism T, equivalent to the statement: for any σ-algebra $\mathfrak{F}_0 \subseteq \mathfrak{F}$

$$T^{-1}\mathfrak{F}_0 \subseteq \mathfrak{F}_0 \implies \mathfrak{F}_0 = T^{-1}\mathfrak{F}_0.$$

Such a condition was found in the works of Adler [18] and Sinai [19], namely: $h(T) = 0$. $\triangleright$

Proposition 1 shows that L_2 coboundaries need not be included in the Gordin space. Theorem 1, Corollary 3, and Theorem 2 below apply to K-automorphisms, which have positive entropy [13, Theorem 18.9], in particular to Bernoulli shifts [13, Proposition 3.51]. Ergodic automorphisms of n-dimensional tori ($n \geqslant 2$) are isomorphic to Bernoulli shifts [20], so have positive entropy, but irrational rotations of the circle have zero entropy [21, p. 252].

In addition, recall that the condition $h(T) = 0$ can also be expressed by the following equality of σ-algebras (see, for example, [13, p. 320]):

$$\Pi(T, \mathfrak{F}) = \mathfrak{F},$$

where $\Pi(T, \mathfrak{F})$ is the Pinsker σ algebra, i. e., $\Pi(T, \mathfrak{F}) = \{A \in \mathfrak{F} : h(\{A, \Omega \setminus A\} = 0)\}$. Thus, for automorphisms with positive entropy, the Pinsker σ-algebra $\Pi(T, \mathfrak{F})$ is a proper σ-subalgebra of the σ-algebra $\mathfrak{F}$. If $\Pi(T, \mathfrak{F}) = \{\varnothing, \Omega\}$, then the automorphism T is called a K-automorphism.

For $\mathfrak{F}_0$ satisfying $T^{-1}\mathfrak{F}_0 \subseteq \mathfrak{F}_0$, let's put

$$\mathfrak{F}_{-\infty} = \bigcap_{n<0} \mathfrak{F}_n, \quad \mathfrak{F}_{+\infty} = \bigvee_{n\geqslant 0} \mathfrak{F}_n.$$

Proposition 2. *Let $h(T) > 0$. The Gordin class $\mathfrak{G}(T, \mathfrak{F}_0)$ is a linear space invariant under the Koopman operator; and it is also dense subset of the first Baire category in $L_2(\Omega, \mathfrak{F}_{+\infty}, \mu) \ominus L_2(\Omega, \mathfrak{F}_{-\infty}, \mu)$.*

$\lhd$ For the Gordin class to be linear, it is sufficient to check that the sum of two functions from the Gordin class is a function from the Gordin class generated by the same σ-algebra. Let $f \in H_{m,n}$ and $g \in H_{p,q}$, i. e.,

$$f = \varphi - E_n\varphi, \quad g = \psi - E_q\psi$$

for some functions $\varphi \in L_2(\Omega, \mathfrak{F}_m, \mu)$ and $\psi \in L_2(\Omega, \mathfrak{F}_p, \mu)$. Then, assuming for definiteness that $q \leqslant n$, we obtain

$$f + g = \varphi + \psi - E_n\varphi - E_q\psi = \xi - E_q\xi,$$

where $\xi = \varphi + \psi - E_n\varphi$. It is clear that

$$E_q\xi = E_q(\varphi + \psi - E_n\varphi) = E_q\varphi + E_q\psi - E_qE_n\varphi = E_q\psi.$$

Thus, it showed that

$$H_{m,n} + H_{p,q} \subset H_{\max\{m,p\},\, \min\{n,q\}}.$$

For the Gordin class to be invariant with respect to the Koopman operator, it is sufficient to show that $TH_{n,m} = H_{n-1,m-1}$. Let $g \in L_2(\Omega, \mathfrak{F}_m, \mu)$, then $g \circ T \in L_2(\Omega, \mathfrak{F}_{m-1}, \mu)$ and

$$(g - E_ng) \circ T = g \circ T - \mathbb{E}(g|\mathfrak{F}_n) \circ T = g \circ T - \mathbb{E}\big(g \circ T|T^{-1}\mathfrak{F}_n\big) = g \circ T - E_{n-1}(g \circ T).$$

Let now $f \in L_2(\Omega, \mathfrak{F}_{+\infty}, \mu) \ominus L_2(\Omega, \mathfrak{F}_{-\infty}, \mu)$. By definition $(E_mf)_{m>0}$ is a martingale and $(E_nf)_{n<0}$ is a reverse martingale. Then the two-parameter family of functions $f_{m,n} = E_mf - E_nE_mf = E_mf - E_nf$ will approximate the function f as $m \to +\infty$ and $n \to -\infty$. This follows from Doob's theorem on the convergence of direct and reversed martingales (see, for example, [13, Theorem 14.26]).

The first Baire category for the Gordin class follows from the fact that it is a countable union of closed spaces $H_{m,n}$. $\rhd$

Theorem 2. *Let $h(T) > 0$. The Gordin space $\mathfrak{G}(T)$ is invariant under the Koopman operator; and it is also dense subset of the first Baire category in $L_2(\Omega, \mathfrak{F}, \mu) \ominus L_2(\Omega, \Pi(T, \mathfrak{F}), \mu)$.*

$\lhd$ Invariance follows from Proposition 2. By the Rokhlin–Sinai theorem (see, for example, [13, Theorem 18.9]) on the characterization of K-automorphisms, there is an extremal σ-subalgebra $\mathfrak{F}_0 \subset \mathfrak{F}$ satisfying $T^{-1}\mathfrak{F}_0 \subseteq \mathfrak{F}_0$, such that

$$\mathfrak{F}_{-\infty} = \Pi(T, \mathfrak{F}), \quad \mathfrak{F}_{+\infty} = \mathfrak{F}.$$

From Proposition 2 it follows that the Gordin class $\mathfrak{G}(T, \mathfrak{F}_0)$ is an everywhere dense subset of $L_2(\Omega, \mathfrak{F}, \mu) \ominus L_2(\Omega, \Pi(T, \mathfrak{F}), \mu)$. On the other hand, for any σ-subalgebra $\mathfrak{F}_0 \subset \mathfrak{F}$ satisfying $T^{-1}\mathfrak{F}_0 \subseteq \mathfrak{F}_0$, from the work of Volný [22, Theorem 2] it follows that

$$L_2(\Omega, \mathfrak{F}_{+\infty}, \mu) \ominus L_2(\Omega, \mathfrak{F}_{-\infty}, \mu) \subset L_2(\Omega, \mathfrak{F}, \mu) \ominus L_2(\Omega, \Pi(T, \mathfrak{F}), \mu).$$

Indeed, let $f = g - \mathbb{E}(g|\mathfrak{F}_{-\infty})$ for $g \in L_2(\Omega, \mathfrak{F}_{+\infty}, \mu)$. Then for any $h \in L_2(\Omega, \mathfrak{F}, \mu)$ we have

$$\big(f, \mathbb{E}(h|\Pi(T, \mathfrak{F}))\big)_{L_2} = \big(\mathbb{E}(f|\Pi(T, \mathfrak{F})), h\big)_{L_2} = 0$$

since

$$\begin{aligned}\mathbb{E}(f|\Pi(T, \mathfrak{F})) &= \mathbb{E}(g|\Pi(T, \mathfrak{F})) - \mathbb{E}(\mathbb{E}(g|\mathfrak{F}_{-\infty})|\Pi(T, \mathfrak{F})) \\ &= \mathbb{E}(g|\Pi(T, \mathfrak{F})) - \mathbb{E}(\mathbb{E}(g|\mathfrak{F}_{-\infty} \vee \Pi(T, \mathfrak{F}))|\Pi(T, \mathfrak{F})) = \mathbb{E}(g|\Pi(T, \mathfrak{F})) - \mathbb{E}(g|\Pi(T, \mathfrak{F})) = 0.\end{aligned}$$

In terminology of Volný the closure $c\ell\mathfrak{G}(T)$ is the set of all difference decomposable functions (see [22, p. 116]).

The first Baire category for $\mathfrak{G}(T)$ follows from [23, Theorem 2] and Corollary 3. $\rhd$

Corollary 5. *If T is a K-authomorphism, then the Gordin space $\mathfrak{G}(T)$ is dense in*

$$L_2^0(\Omega, \mathfrak{F}, \mu) := \left\{ f \in L_2(\Omega, \mathfrak{F}, \mu) : \int f\, d\mu = 0 \right\}.$$

It shows that for a K-automorphism on a standard probability space, the Gordin space is not closed. Otherwise we have, by Theorem 1, a uniform rate of convergence in the mean ergodic theorem, contradicting the well-known Krengel's result on arbitrary slow convergence in the mean ergodic theorem.

In conclusion, we note that it would be interesting to find a natural dense class of functions for which the rate of convergence of ergodic averagess for an abstract automorphism with zero entropy can be calculated.

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Ivan V. Podvigin
Sobolev Institute of Mathematics of the Siberian Branch of the RAS,
4 Ac. Koptug Ave., Novosibirsk 630090, Russia,
Senior Researcher
E-mail: `ipodvigin@math.nsc.ru`

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О СКОРОСТИ СХОДИМОСТИ ЭРГОДИЧЕСКИХ СРЕДНИХ ДЛЯ ФУНКЦИЙ ИЗ ПРОСТРАНСТВА ГОРДИНА

Подвигин И. В.[1]

[1] Институт математики им. С. Л. Соболева Сибирского отделения РАН,
Россия, 630090, Новосибирск, проспект ак. Коптюга, 4

E-mail: ipodvigin@math.nsc.ru

Аннотация. Для автоморфизмов с ненулевой энтропией рассмотрен естественный класс функций, названный пространством Гордина. Это пространство есть линейная оболочка классов Гордина, построенных по некоторой инвариантной относительно автоморфизма фильтрации σ-алгебр $\mathfrak{F}_n$. Функция из класса Гордина представляет собой ортогональную проекцию относительно оператора $I - E(f|\mathfrak{F}_n)$ некоторой $\mathfrak{F}_m$-измеримой функции. После работы Гордина о применении мартингального метода для доказательства центральной предельной теоремы, эта конструкция получила свое развитие в работах Далибора Волны. В этой обзорной статье мы рассматриваем эту конструкцию в эргодической теории. Показано, что скорость сходимости эргодических средних в L_2 норме для функций из пространства Гордина просто вычисляется и равна $\mathscr{O}(\frac{1}{\sqrt{n}})$. Также показано, что пространства Гордина есть плотное множество первой категории по Бэру в $L_2(\Omega, \mathfrak{F}, \mu) \ominus L_2(\Omega, \Pi(T, \mathfrak{F}), \mu)$, где $\Pi(T, \mathfrak{F})$ — σ-алгебра Пинскера.

Ключевые слова: скорости сходимости в эргодических теоремах, фильтрация, мартингальный метод.

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