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METRIC CHARACTERISTICS OF CLASSES OF COMPACT SETS ON CARNOT GROUPS WITH SUB-LORENTZIAN STRUCTURE#

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Abstract. We consider classes of mappings of Carnot groups that are intrinsically Lipschitz and defined on compact subsets, and describe the metric characteristics of their images under the condition that a sub-Lorentzian structure is introduced on the image. This structure is a sub-Riemannian generalization of Minkowski geometry. One of its features is the unlimitedness of the balls constructed with respect to the intrinsic distance. In sub-Lorentzian geometry, the study of spacelike surfaces whose intersections with such balls are limited, is of independent interest. If the mapping is defined on an open set, then the formulation of space-likeness criterion reduces to considering the connectivity component of the intersection containing the center of the ball and analyzing the properties of the sub-Riemannian differential matrix. If the domain of definition of the mapping is not an open set, then the question arises what conditions can be set on the mapping that guarantee the boundedness of the intersection of the image of a compact set with a sub-Lorentzian ball. In this article, this problem is resolved: we consider that part of the intersection that can be parameterized by the connectivity component of the intersection of the image of the sub-Riemannian differential and the ball. In addition, using such local parameterizations, a set function is introduced, which is constructed similarly to Hausdorff measure. We show that this set function is also a measure. As an application, the sub-Lorentzian area formula is established.

Keywords: Carnot group, Lipschitz mapping, compact set, sub-Lorentzian structure, quasi-additive set function, area formula.

AMS Subject Classification: 28A75, 28A15.

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1. Preliminaries

The article is devoted to research in the field of geometric measure theory on sub-Lorentzian structures. These structures are a nonholonomic generalization of Minkowski geometry (see [1] and references therein). Research into both the structures themselves and their applications in physics [2, 3] began relatively recently. Article [4] is one of the first works in which such structures were studied. For further acquaintance with recent results established for sub-Lorentzian structures and their generalizations (for example, in the case of a multidimensional timelike coordinate), see, e. g., [5] and the list of cited literature.

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Definition 1 (see., e. g., [6]). A *Carnot group* is a connected simply-connected Lie group $\mathbb{G}$ such that its Lie algebra V is graded, i. e., is represented in the form

$$V = \bigoplus_{j=1}^{M} V_j, \quad [V_1, V_j] = V_{j+1}, \quad j < M, \quad [V_1, V_M] = \{0\}.$$

Denote the dimension of V_k (at each x) by $\dim V_k$. If $X_j \in V_k$ then the number k is called the *degree* of the field X_j and is denoted by $\deg X_j$.

Definition 2. Let $\mathbb{G}$ be a Carnot group of the topological dimension N, and $w = \exp\big(\sum_{i=1}^{N} w_i X_i\big)(v)$. Define the value d_2 as

$$d_2(v, w) = \max\left\{ \Bigg(\sum_{j:\,\deg X_j = 1} w_j^2\Bigg)^{\frac{1}{2}}, \Bigg(\sum_{j:\,\deg X_j = 2} w_j^2\Bigg)^{\frac{1}{2\cdot 2}}, \dots, \Bigg(\sum_{j:\,\deg X_j = M} w_j^2\Bigg)^{\frac{1}{2\cdot M}} \right\}.$$

The set $\{w \in \mathbb{G} : d_2(v, w) < r\}$ is called a *ball with respect to d_2 of the radius $r > 0$ centered at* v, and is denoted by $\mathrm{Box}_2(v, r)$.

By the formulas of the group operation, which follow from the Baker–Campbell–Haussdorf formula, direct calculations imply that the Hausdorff dimension of the group $\mathbb{G}$ with respect to d_2 equals

$$\nu = \sum_{j=1}^{M} j \dim V_j.$$

Definition 3. Define a set function $\mathscr{H}^\nu$ for $A \subset \mathbb{G}$ as

$$\mathscr{H}^\nu(A) = \prod_{k=1}^{M} \omega_{\dim V_k} \cdot \lim_{\delta \to 0} \inf \left\{ \sum_{i \in \mathbb{N}} r_i^\nu : \bigcup_{i \in \mathbb{N}} \mathrm{Box}_2(y_i, r_i) \supset A,\ y_i \in A,\ r_i < \delta \right\},$$

where the infimum is taken over all coverings of A.

To this end, the symbol ω_l stands for a volume of a Euclidean ball of the unit radius in $\mathbb{R}^l$.

The set function $\mathscr{H}^\nu$ is a measure; it follows from the quasi-additivity of $\mathscr{H}^\nu$ and results of [7, 8]. It is also easy to prove that $\mathscr{H}^\nu$ and $\mathscr{H}^N$ are absolutely continuous one with respect to another, and are doubling. The derivative of $\mathscr{H}^N$ with respect to $\mathscr{H}^\nu$ at $x \in \mathbb{G}$ equals $\sqrt{\det(g(x))}$, where g is a Riemann tensor on $\mathbb{G}$ defined by the basis vector fields $X_1, \dots, X_N$ (note that $\mathbb{G}$ with its basis vector fields can also be considered as a Riemannian manifold).

2. Sub-Riemannian Differentiability and Sub-Lorentzian Structure

Consider one more Carnot group $\widetilde{\mathbb{G}}$ with Lie algebra $\widetilde{V} = \bigoplus_{k=1}^{\widetilde{M}} \widetilde{V}_k$ and basis vector fields $\widetilde{X}_1, \dots, \widetilde{X}_{\widetilde{N}}$. Denote the quasimetric constructed on $\widetilde{\mathbb{G}}$ in the same way as in Definition 2 (with obvious changes), by $\widetilde{d}_2$.

Let us pass to the sub-Riemannian analogue of differentiability for our case, and to some important results.

Definition 4 ([9]; see also [10]). Let $\mathbb{G}$ and $\widetilde{\mathbb{G}}$ be Carnot groups, $\Omega \subset \mathbb{G}$, and $\varphi : \Omega \to \widetilde{\mathbb{G}}$. The mapping φ is *hc-differentiable*, or *differentiable in the sub-Riemannian sense*, at the (limit) point $x \in \Omega$ if there exists a horizontal homomorphism $\mathscr{L}_x : \mathbb{G} \to \widetilde{\mathbb{G}}$ such that

$$\widetilde{d}_2\big(\varphi(y), \mathscr{L}_x\langle y\rangle\big) = o(1) \cdot d_2(x, y), \quad \text{where} \quad o(1) \to 0 \ \text{ if } \ \Omega \ni y \to x. \tag{1}$$

The *hc-differential* (or *sub-Riemannian differential*) $\mathscr{L}_x$ is denoted by $\widehat{D}\varphi(x)$.

Definition 5. Let $\mathbb{G}$ and $\widetilde{\mathbb{G}}$ be Carnot groups, $\Omega \subset \mathbb{G}$, and $\varphi : \Omega \to \widetilde{\mathbb{G}}$. If φ is a Lipschitz mapping with respect to quasimetrics d_2 and $\widetilde{d}_2$ then φ is *Lipschitz in the intrinsic sense.*

The next result was obtained for the first time by P. Pansu [9] for open sets, and by S. K. Vodopyanov ([11]; see also, e. g., [10], where it is established for a general case of Carnot–Carathéodory spaces) for measurable sets.

Theorem 1. *Let $\mathbb{G}$ and $\widetilde{\mathbb{G}}$ be Carnot groups, $E \subset \mathbb{G}$ be a measurable set, and $\varphi : E \to \widetilde{\mathbb{G}}$ be a Lipschitz mapping in the intrinsic sense. Then it is hc-differentiable almost everywhere. Moreover, $\widehat{D}\varphi$ consists of the "diagonal" $(\dim \widetilde{V}_k \times \dim V_k)$-blocks, while all other elements vanish.*

Note 1 (see, e. g., [12]). For arbitrary $\varepsilon > 0$, there exists a set Σ such that $\mathscr{H}^\nu(\Sigma) < \varepsilon$ and on $E \setminus \Sigma$ the value $o(1)$ from (1) is uniform.

To this end, by "diagonal" ones, we mean the blocks consisting of elements such that their lines' numbers correspond to the fields from $\widetilde{V}_k$, and columns' numbers, to the fields from V_k, $k = 1, \dots, M$. Thus, the dimension of kth block is $\dim \widetilde{V}_k \times \dim V_k$, $k = 1, \dots, M$. Denote "diagonal" $(\dim \widetilde{V}_k \times \dim V_k)$-blocks constituting the matrix of $\widehat{D}\varphi$, by $\widehat{D}_k\varphi$, $k = 1, \dots, M$.

In the paper, we assume that $D \subset \mathbb{G}$ is compact, it possesses the properties of the set $E \setminus \Sigma$ from Note 1, the mapping φ is continuously hc-differentiable in the topology of its domain, also, for $\mathbb{G}$ and $\widetilde{\mathbb{G}}$ we have $\widetilde{M} \geqslant M$, and for at least one $k_0 \in \overline{[1, M]}$ we have $\dim \widetilde{V}_{k_0} > \dim V_{k_0}$, and $\dim \widetilde{V}_k \geqslant \dim V_k$ for all the other $k \neq k_0$. Then, the topological dimension $\widetilde{N}$ of $\widetilde{\mathbb{G}}$ is strictly greater than N.

For each $k = 1, \dots, M$, choose integers $\dim \widetilde{V}_k^- \in \overline{[0, \dim \widetilde{V}_k - \dim V_k]}$. Set $\widetilde{n}_0 = 0$ and $\widetilde{n}_k = \sum_{l=1}^k \dim \widetilde{V}_l$.

Definition 6. Let $w = \exp\big(\sum_{i=1}^{\widetilde{N}} w_i \widetilde{X}_i\big)(v)$. Put $\mathfrak{d}_2^2(v, w)$ be equal to

$$\max_{k=1,\dots,\widetilde{M}} \left\{ \left| \sum_{j=\widetilde{n}_{k-1}+\dim \widetilde{V}_k^- +1}^{\widetilde{n}_k} w_j^2 - \sum_{j=\widetilde{n}_{k-1}+1}^{\widetilde{n}_{k-1}+\dim \widetilde{V}_k^-} w_j^2 \right|^{\frac{1}{k}} \cdot \operatorname{sgn} \left(\sum_{j=\widetilde{n}_{k-1}+\dim \widetilde{V}_k^- +1}^{\widetilde{n}_k} w_j^2 - \sum_{j=\widetilde{n}_{k-1}+1}^{\widetilde{n}_{k-1}+\dim \widetilde{V}_k^-} w_j^2 \right) \right\}.$$

The set $\{w \in \widetilde{\mathbb{G}} : \mathfrak{d}_2^2(v, w) < r^2\}$ is called the *ball in $\mathfrak{d}_2^2$ of the radius $r > 0$ centered at v* and is denoted by $\operatorname{Box}_{\mathfrak{d}}(v, r)$.

3. Sub-Lorentzian Analogs of Distance and Measure

To study the metric properties of surfaces lying in $\widetilde{\mathbb{G}}$, it is enough to consider the above analogue of the squared distance $\mathfrak{d}_2^2$, without considering the roots of the quantities involved in the definition. Let us first describe a measure constructed with a system of these balls on classes of images of open sets lying in $\widetilde{\mathbb{G}}$. In [5], the measure of the image $B \subset \widetilde{\mathbb{G}}$ of an open set from $\mathbb{G}$ is defined as

$$\omega_{\mathfrak{d}} \cdot \lim_{\delta \to 0} \inf \left\{ \sum_{i \in \mathbb{N}} r_i^\nu : \bigcup_{i \in \mathbb{N}} \big(\operatorname{Box}_{\mathfrak{d}}(x_i, r_i) \cap^{x_i} B\big) \supset B,\ x_i \in B,\ r_i < \delta \right\},$$

where the infimum is taken over all coverings of B, $\omega_{\mathfrak{d}} = \prod_{k=1}^M \omega_{\dim V_k}$, and $\operatorname{Box}_{\mathfrak{d}}(x_i, r_i) \cap^{x_i} B$ denotes a connectivity component of $\operatorname{Box}_{\mathfrak{d}}(x_i, r_i) \cap B$ containing x_i. The necessity of using $\operatorname{Box}_{\mathfrak{d}}(x_i, r_i) \cap^{x_i} B$ instead of $\operatorname{Box}_{\mathfrak{d}}(x_i, r_i) \cap B$ relies to the specifics of the sub-Lorentzian structure, that is, a non-boundedness of balls. See details and comments in [5, 13].

In our case, we consider the image of a compact set, which is not open in the general case. Therefore, consideration of the connectivity component does not make sense, and it needs to be replaced with another object, which is the image of a subset of some open connected set. To define a new set function and prove its correctness, we first of all describe the class of mappings φ under study and derive the main properties of its sub-Riemannian differential.

In each "diagonal" $(\dim \widetilde{V}_k \times \dim V_k)$-block $\widehat{D}_k$ of the sub-Riemannian differential $\widehat{D}\varphi$, denote the part consisting of first $\dim V_k^-$ lines, by $\widehat{D}_k^-$, and the rest, by $\widehat{D}_k^+$, $k = 1, \dots, M$. The following is true.

Theorem 2. *Fix k such that the rank of $\widehat{D}_k^+$ equals $r_k \leqslant \dim V_k$, $r_k > 0$, and orthogonal mapping O_k that transfers lines of $\widehat{D}_k$ to $\mathbb{R}^{r_k} \times 0^{\dim V_k - r_k}$.*

Assume that for r_k independent lines of $[\widehat{D}_k^+ O_k]$ with numbers $i_1, \dots, i_{r_k}$ constituting the matrix $\widehat{L}_{r_k}^+$, the following holds: lengths of columns of $\big[\widehat{D}_k^- O_k\big]\big(\widehat{L}_{r_k}^+\big)^{-1}$ do not exceed $1/r_k - c$, $c > 0$. Then the validity of this property for a matrix composed of lines with numbers $i_1, \dots, i_{r_k}$ does not depend on the choice of O_k.

Moreover, if $\widehat{L}_{r_k}^+$ exists for some collection of lines and some O_k, then the validity of the estimate of lines' lengths fails if a single line from $\widehat{L}_{r_k}^+$ is replaced by a line from $\big[\widehat{D}_k^- O_k\big]$.

Here $[Q]$ denotes a part of a matrix Q consisting of its first r_k columns.

To this end, we will assume that the matrix of $\widehat{D}\varphi$ enjoys conditions of Theorem 2. Moreover, suppose that on $D \setminus D_0$, the mapping φ is bijective on its image, where $D_0 = \big\{y \in D : \operatorname{rank} \widehat{D}\varphi(y) < N\big\}$.

For images of every mapping $w \mapsto \widehat{D}\varphi(y)\langle w\rangle$, $y \in D$, the next theorem is true [5].

Theorem 3 (see also [5, Theorem 2.1])**.** *Fix $y \in D$ and $x = \varphi(y)$. Then, if the above assumptions on φ hold, the following is true.*

1. *The intersection of $\big\{w \in \widetilde{\mathbb{G}} : \mathfrak{d}_2^2(x, w) \leqslant 0\big\}$ and $\operatorname{Im} \widehat{D}\varphi(y)$ consists of a single point x.*

2. *The intersection of a ball $\operatorname{Box}_{\mathfrak{d}}(x, r)$ with $\operatorname{Im} \widehat{D}\varphi(y)$ is bounded.*

3. *On $\operatorname{Im} \widehat{D}\varphi(y)$ (for each fixed y), the values $(\widetilde{d}_2)^2$ and $\mathfrak{d}_2^2$ are locally bi-Lipschitz equivalent.*

Let us pass to the description of the measure on the images of compact sets. We will construct it by analogy with the Hausdorff measure described in Definition 3. As stated earlier, the main idea is to parameterize by a subset of some open connected set the fragment of intersection of the image of the mapping with a ball in $\mathfrak{d}_2^2$, which is part of the covering of the image of the mapping. In addition, in contrast to [5], the degeneracy of the sub-Riemannian differential is allowed. For this reason, and also due to the specificity of the parameterization of intersections, we introduce additional restrictions on the radii of not only the covering balls, but also their preimages, and show that even under such restrictions the set of coverings is non-empty.

Fix $\varepsilon > 0$, then [10] there exists such $\delta = \delta(\varepsilon) > 0$ that if $y, v \in D$, and $d_2(y, w) < \delta$ then

$$\widetilde{d}_2\big(\widehat{D}\varphi(y)\langle w\rangle, \varphi(w)\big) < \varepsilon d_2(y, w).$$

As parameterizing sets, consider the images of sub-Riemannian differentials of the mapping, which are analogues of tangent spaces. Then the preimages of the intersections of such "tangent planes" and $\mathfrak{d}_2^2$-balls must be small enough to ensure that the quantity $o(1)$ from the definition of sub-Riemannian differentiability is small. For each $x = \varphi(y)$, $y \in D \setminus D_0$, $x \notin \varphi(D_0)$, and $\varepsilon > 0$, denote by the symbol $r_{x,\varepsilon}$ the value

$$\sup\big\{r > 0 : \ \widehat{D}\varphi(y)^{-1}\big\langle \operatorname{Box}_{\mathfrak{d}}(x, r) \cap^x \operatorname{Im} \widehat{D}\varphi(y)\big\rangle \subset \operatorname{Box}_2\big(y, \min\{\delta(\varepsilon), \varepsilon\}\big)\big\}.$$

Also, for $y=\varphi^{-1}(x)$, where $y\in D\setminus D_0$ and $\varphi(y)\notin\varphi(D_0)$, put $r_{y,\varepsilon}=r_{x,\varepsilon}$. If $y\in D_0$ or $\varphi(y)\in\varphi(D_0)$ then $r_{y,\varepsilon}$ is equal to

$$\sup\bigl\{r>0:\ \widehat{D}\varphi(y)^{-1}\bigl\langle \operatorname{Box}_{\mathfrak{d}}(x,r)\cap^x \operatorname{Im}\widehat{D}\varphi(y)\bigr\rangle\cap\bigl(\ker\widehat{D}\varphi(y)\bigr)^{\perp}\subset \operatorname{Box}_2\bigl(y,\min\{\delta(\varepsilon),\varepsilon\}\bigr)\bigr\}$$

(if $y\notin D_0$ and $\varphi(y)\in\varphi(D_0)$ then $(\ker\widehat{D}\varphi(y))^{\perp}=\mathbb{G}$). Set $r_{x,\varepsilon}=\sup_{y:\varphi(y)=x}\{r_{y,\varepsilon}\}$.

By Theorem 3, the values $r_{x,\varepsilon}$, $r_{y,\varepsilon}$ are positive for all $\varepsilon>0$, $x\in\varphi(D)$, $y\in D$. This property follows from the boundedness of the intersections of images of the sub-Rienammian differentials $\operatorname{Im}\widehat{D}\varphi(y)$ with balls $\operatorname{Box}_{\mathfrak{d}}(x,r)$, $y\in D$, $x=\varphi(y)$.

Let us describe one more value, that is, maximal possible radius of a sub-Riemannian ball containing the preimage of the intersection of a "tangent plane" and $\mathfrak{d}_2^2$-ball. For y and $\varepsilon>0$ and $r\leqslant r_{y,\varepsilon}$, and define $\widehat{r}_{r,y,\varepsilon}$ as

$$\inf\bigl\{\widehat{r}>0:\ \widehat{D}\varphi(y)^{-1}\bigl\langle \operatorname{Box}_{\mathfrak{d}}(x,r)\cap^x \operatorname{Im}\widehat{D}\varphi(y)\bigr\rangle\cap\bigl(\ker\widehat{D}\varphi(y)\bigr)^{\perp}\subset \operatorname{Box}_2(y,\widehat{r})\bigr\}.$$

If $\widehat{D}\varphi(y)$ is degenerate then we consider the intersection with the orthogonal complement of its kernel since otherwise such preimage is not bounded. If $\widehat{D}\varphi(y)=0$ then put

$$\widehat{r}_{r,y,\varepsilon}=\min\biggl\{\max\biggl\{1,\frac{1}{\operatorname{Lip}(\widehat{D}\varphi)}\biggr\}r,\ \delta(\varepsilon),\ \varepsilon\biggr\},$$

where $\operatorname{Lip}(\widehat{D}\varphi)=\sup_{y,w}\widetilde{d}_2(\varphi(y),\widehat{D}\varphi(y)\langle w\rangle)$, $y\in D$, $d_2(w,y)=1$. Since $\widehat{D}\varphi$ is continuous on the compact set D, we have $\operatorname{Lip}(\widehat{D}\varphi)<\infty$.

Fix $x\in\varphi(D)$, $x=\varphi(y)$, and consider $\operatorname{Box}_{\mathfrak{d}}(x,r)\cap^x\operatorname{Im}\widehat{D}\varphi(y)$, $r>0$. Let π_x be a projection $\varphi(w)\mapsto\widehat{D}\varphi(y)\langle w\rangle$, where $\varphi(y)=x$. For $A\subset\varphi(D)$, $x\in A$, $\varepsilon>0$, and $r<r_{x,\varepsilon}$, denote by $\operatorname{Box}_{\mathfrak{d}}(x,r)\widehat{\cap}^x A$ the set

$$\begin{cases}\pi_x^{-1}\bigl(\operatorname{Box}_{\mathfrak{d}}(x,r)\cap^x\operatorname{Im}\widehat{D}\varphi(y)\bigr), & \text{if } \operatorname{rank}\widehat{D}\varphi(y)=N \text{ and } x\notin\varphi(D_0),\\ \bigcup\limits_{y:\,\varphi(y)=x}\pi_{r,y,\varepsilon}^{-1}\bigl(\operatorname{Box}_{\mathfrak{d}}(x,\min\{r,r_{y,\varepsilon}\})\cap^x\operatorname{Im}\widehat{D}\varphi(y)\bigr), & \text{if } \operatorname{rank}\widehat{D}\varphi(y)<N \text{ и } x\notin\varphi(D\setminus D_0),\\ \mathscr{A}, & \text{if } x\in\varphi(D\setminus D_0)\cap\varphi(D_0),\end{cases}$$

where $\pi_{r,y,\varepsilon}$ equals the mapping

$$\begin{cases}\varphi(w)\mapsto\widehat{D}\varphi(y)\langle w\rangle, & \text{if } w\in\operatorname{Box}_2(y,\widehat{r}_{\rho,y,\varepsilon}),\ \rho=\min\{r,r_{y,\varepsilon}\}, \text{ and } \widehat{D}\varphi(y)\neq 0,\\ \varphi(w)\mapsto\widehat{D}\varphi(y)\langle w\rangle, & \text{if } w\in\operatorname{Box}_2(y,\widehat{r}_{r,y,\varepsilon}) \text{ and } \widehat{D}\varphi(y)=0,\\ \mathbf{x},\ \mathbf{x}\notin\varphi(D), & \text{otherwise,}\end{cases}$$

$\mathbf{x}\notin\varphi(D)$ is a fixed point, and $\mathscr{A}$ is the union of the sets

$$\pi_x^{-1}\bigl(\operatorname{Box}_{\mathfrak{d}}(x,r)\cap^x\operatorname{Im}\widehat{D}\varphi(y)\bigr)$$

and

$$\bigcup_{y:\,\varphi(y)=x}\pi_{r,y,\varepsilon}^{-1}\bigl(\operatorname{Box}_{\mathfrak{d}}(x,\min\{r,r_{y,\varepsilon}\})\cap^x\operatorname{Im}\widehat{D}\varphi(y)\bigr).$$

Note that $\pi_{r,y,\varepsilon}^{-1}$ is not a mapping in the general case, since it may assign to one element $\widehat{D}\varphi(y)\langle w\rangle$ several elements $w'\in\operatorname{Box}_2(y,\delta(y))$ such that $\widehat{D}\varphi(y)\langle w'\rangle=\widehat{D}\varphi(y)\langle w\rangle$, and $\varphi(w')\neq\varphi(w)$. We use $\pi_{r,y,\varepsilon}$ instead of π_x due to necessity of boundedness of intersections $\operatorname{Box}_{\mathfrak{d}}(x,r)\cap\varphi(D_0)$ and their preimages.

DEFINITION 7. Assume that $A \subset \widetilde{\mathbb{G}}$ is a subset of the image of a compact set $D \subset \mathbb{G}$ under Lipschitz in the intrinsic sense mapping φ which is hc-differentable everywhere on its domain. Suppose that on $D \setminus D_0$, the mapping φ is bijective on its image. Define $\mathscr{H}^{\nu}_{\mathfrak{d}}(A)$ as

$$\prod_{k=1}^{M} \omega_{\dim V_k} \cdot \liminf_{\varepsilon \to 0} \Big\{ \sum_{i \in \mathbb{N}} \delta_i r_i^{\nu} : \bigcup_{i \in \mathbb{N}} \big(\mathrm{Box}_{\mathfrak{d}}(x_i, r_i) \widehat{\cap}^{x_i} A \big) \supset A,\, x_i \in A,$$
$$r_i < \min\{\varepsilon, \delta(\varepsilon), r_{x_i,\varepsilon}\},\, \delta_i = \varepsilon \text{ if } \varphi^{-1}(x_i) \cap D_0 \neq \varnothing \text{ and } \delta_i = 1 \text{ if } x_i \notin \varphi(D_0) \Big\},$$

where the infimum is taken over all coverings $\bigcup_{i \in \mathbb{N}} \big(\mathrm{Box}_{\mathfrak{d}}(x_i, r_i) \widehat{\cap}^{x_i} A \big) \supset A$ of A.

4. Main Results

Since the goal of the paper is to describe the metric characteristics of subsets of $\varphi(D)$, it is necessary to study the properties of the set function $\mathscr{H}^{\nu}_{\mathfrak{d}}$. In particular, we need to show that it is defined correctly (that is, the class of coverings from Definition 7 is non-empty) and is a measure for the class of mappings under study. To establish this, we prove the following properties.

Lemma 1. *The set function*

$$\mathbb{G} \supset A \mapsto \mathscr{H}^{\nu}_{\mathfrak{d}}(\varphi(A)) \tag{2}$$

is defined correctly for images of sets of zero $\mathscr{H}^{\nu}$-measure, and it is absolutely continuous with respect to $\mathscr{H}^{\nu}$ on $\mathbb{G}$.

A stronger statement is also true.

Theorem 4. *The value $\mathscr{H}^{\nu}_{\mathfrak{d}}$ is defined correctly for images of measurable sets $E \subset D$ in the sense that the set of coverings from the definition of $\mathscr{H}^{\nu}_{\mathfrak{d}}(\varphi(E))$ is not empty for each $\varepsilon > 0$.*

Moreover, there exists such $T = T(D) < \infty$ that $\mathscr{H}^{\nu}_{\mathfrak{d}}(\varphi(E)) < T\mathscr{H}^{\nu}(E)$, where $T < \infty$ and it depends on $\mathbb{G}$ and φ only.

It is not hard to prove the result on the degeneration set.

Lemma 2. *We have $\mathscr{H}^{\nu}_{\mathfrak{d}}(\varphi(D_0)) = 0$.*

Despite the obviousness of the statement about the image of a degeneracy set, establishing that for the image of an arbitrary measurable set the sum of terms with coefficients tending to zero (of the form $\delta_i = \varepsilon$) from Definition 7 is small is a non-trivial task. The difficulty is that we are looking for an infimum on the sums. For this reason, it seems natural to conclude that the more balls with centers at the points of $\varphi(D_0)$ and, accordingly, $\delta_i = \varepsilon$ cover the set, the lower the value of the sum we achieve. The next theorem shows that, in fact, this is not the case, and the sum of terms with $\delta_i = \varepsilon$ can be arbitrarily small for values close to the infimum.

Theorem 5. *Consider a closed set $\widehat{B}$, its image $B = \varphi(\widehat{B} \cap D)$, and $\sigma > 0$. Suppose that $\varepsilon_0 > 0$ is such that for each $\varepsilon < \varepsilon_0$, the value $\mathscr{H}^{\nu}_{\mathfrak{d},\varepsilon}(B)$ being equal to*

$$\prod_{k=1}^{M} \omega_{\dim V_k} \cdot \inf \Big\{ \sum_{i \in \mathbb{N}} \delta_i r_i^{\nu} : \bigcup_{i \in \mathbb{N}} \big(\mathrm{Box}_{\mathfrak{d}}(x_i, r_i) \widehat{\cap}^{x_i} B \big) \supset B,\, x_i \in B,$$
$$r_i < \min\{\varepsilon, \delta(\varepsilon), r_{x_i,\varepsilon}\},\, \delta_i = \varepsilon \textit{ if } \varphi^{-1}(x_i) \cap D_0 \neq \varnothing \textit{ and } \delta_i = 1 \textit{ if } x_i \notin \varphi(D_0) \Big\},$$

differs from $\mathscr{H}^\nu_{\mathfrak{d}}(B)$ *not more than by* $\sigma/4$. *Then, for sufficiently small* $\xi > 0$, $\xi < \varepsilon_0$, *and coverings*

$$\bigcup_{i\in\mathbb{N}} \bigl(\operatorname{Box}_{\mathfrak{d}}(x_i,r_i)\widehat{\cap}^{x_i}B\bigr) \supset B, \quad r_i < \min\{\xi, \delta(\xi), r_{x_i,\xi}\}$$

such that

$$\left|\prod_{k=1}^{M}\omega_{\dim V_k}\cdot\sum_{i\in\mathbb{N}}\delta_i r_i^\nu - \mathscr{H}^\nu_{\mathfrak{d},\xi}(B)\right| < \frac{\sigma}{4},$$

the estimate

$$\xi\cdot\prod_{k=1}^{M}\omega_{\dim V_k}\cdot\sum_{i:\, x_i\in\varphi(D_0)} r_i^\nu < \sigma$$

holds.

This statement is one of key results in the proof of the quasi-additivity of the set function (2).

Definition 8 (see, e. g., [7, 8]). A set function Φ is called *quasi-additive* if for any finite collection of pairwise disjoint open balls $\{B_j\}_{j=1}^J$ lying in some open ball B_0, the inequality

$$\sum_{j=1}^{J}\Phi(B_j) \leqslant \Phi(B_0)$$

holds.

Theorem 6. *The set function* (2) *is quasi-additive.*

The above results together with [7, 8] imply the following property.

Theorem 7. *The function* (2) *is differentiable almost everywhere with respect to the measure* $\mathscr{H}^\nu$: *for almost all* $x\in D$ *the limit*

$$\Phi'(y) = \lim_{r\to 0}\frac{\mathscr{H}^\nu_{\mathfrak{d}}(\varphi(\operatorname{Box}_2(y,r)))}{\mathscr{H}^\nu(\operatorname{Box}_2(y,r))} = D_{\mathscr{H}^\nu}\mathscr{H}^\nu_{\mathfrak{d}}(y)$$

exists. Moreover, the function (2) *is recoverable by its derivative: for* $A\subset D$, *we have*

$$\mathscr{H}^\nu_{\mathfrak{d}}(\varphi(A)) = \int\limits_A D_{\mathscr{H}^\nu}\mathscr{H}^\nu_{\mathfrak{d}}(y)\, d\mathscr{H}^\nu(y).$$

Finally, we deduce the area formula.

Theorem 8. *Let* $\mathbb{G}$ *and* $\widetilde{\mathbb{G}}$ *be Carnot groups,* $D\subset\mathbb{G}$ *is a compact set, and the mapping* $\varphi: D\to\widetilde{\mathbb{G}}$ *is continuously hc-differentiable in the tolopology of its domain, and the value* $o(1)$ *from* (1) *is uniform on* D. *Assume that* φ *is bijective on its image on* $D\setminus D_0$, *and* $\widehat{D}\varphi$ *enjoys conditions of Theorem* 2 *everywhere. Suppose also that* $\widetilde{M}\geqslant M$, *and for at least one* $k_0\in\overline{[1,M]}$ *the inequality* $\dim\widetilde{V}_{k_0} > \dim V_{k_0}$ *holds, and* $\dim\widetilde{V}_k\geqslant\dim V_k$ *for all other* $k\neq k_0$. *Then*

$$\Phi'(y) = \prod_{k=1}^{M}\sqrt{\det\bigl(\widehat{D}_k^+\varphi(y)^*\widehat{D}_k^+\varphi(y) - \widehat{D}_k^-\varphi(y)^*\widehat{D}_k^-\varphi(y)\bigr)}$$

for almost all $y\in D\setminus\varphi^{-1}(\varphi(D_0))$, *and the area formula*

$$\int\limits_{A\cap D}\prod_{k=1}^{M}\sqrt{\det\bigl(\widehat{D}_k^+\varphi(y)^*\widehat{D}_k^+\varphi(y) - \widehat{D}_k^-\varphi(y)^*\widehat{D}_k^-\varphi(y)\bigr)}\, d\mathscr{H}^\nu(y) = \mathscr{H}^\nu_{\mathfrak{d}}(\varphi(A))$$

holds.

NOTE 2. If $E \subset \mathbb{G}$ is a measurable set of finite measure, $\varphi : E \to \widetilde{\mathbb{G}}$ is continuously hc-differentiable almost everywhere, then for arbitrary $\varepsilon > 0$, there exists a compact set $D \subset E$ such that $\mathscr{H}^\nu(E \setminus D) < \varepsilon$, and on D the value $o(1)$ from (1) is uniform; thus, the set D satisfies the conditions of Theorem 8.

References

1. *Miklyukov, V. M., Klyachin, A. A. and Klyachin, V. A. Maksimal'nye Poverkhnosti v Prostranstve-Vremeni Minkovskogo* [Maximal Surfaces in Minkowski Space-Time], Volgograd, VolSU, 2011, 530 p. (in Russian).
2. *Krym, V. R. and Petrov, N. N.* Equations of Motion of a Charged Particle in a Five-Dimensional Model of the General Theory of Relativity with a Nonholonomic Four-Dimensional Velocity Space, *Vestnik St. Petersburg University: Mathematics*, 2007, vol. 40, no. 1, pp. 52–60. DOI: 10.3103/S1063454107010062.
3. *Krym, V. R. and Petrov, N. N.* The Curvature Tensor and the Einstein Equations for a Four-Dimensional Nonholonomic Distribution, *Vestnik St. Petersburg University: Mathematics*, 2008, vol. 41, no. 3, pp. 256–265. DOI: 10.3103/S1063454108030060.
4. *Berestovskii, V. N. and Gichev, V. M.* Metrized Left-Invariant Orders on Topological Groups, *St. Petersburg Mathematical Journal*, 2000, vol. 11, no. 4, pp. 543–565.
5. *Karmanova, M. B.* Lipschitz Images of Open Sets on Sub-Lorentzian Structures, *Siberian Advances in Mathematics*, 2024, vol. 34, no. 1, pp. 67–79. DOI: 10.1134/S1055134424010036.
6. *Folland, G. B. and Stein, E. M. Hardy Spaces on Homogeneous Group*, Princeton, Princeton University Press, 1982, 286 p.
7. *Vodopyanov, S. K. and Ukhlov, A. D.* Set Functions and Their Applications in the Theory of Lebesgue and Sobolev Spaces. I, *Siberian Advances in Mathematics*, 2004, vol. 14, no. 4, pp. 78–125.
8. *Vodopyanov, S. K. and Ukhlov, A. D.* Set Functions and Their Applications in the Theory of Lebesgue and Sobolev Spaces. II, *Siberian Advances in Mathematics*, 2005, vol. 15, no. 1, pp. 91–125.
9. *Pansu, P.* Metriques de Carnot–Caratheodory et Quasi-Isometries des Espaces Symetriques de Rang 1, *The Annals of Mathematics*, 1989, vol. 129, no. 1, pp. 1–60. DOI: 10.2307/1971484.
10. *Vodopyanov, S.* Geometry of Carnot–Carathéodory Spaces and Differentiability of Mappings, *The Interaction of Analysis and Geometry. Contemporary Mathematics*, vol. 424, Providence, RI, AMS, 2007, pp. 247–301.
11. *Vodop'yanov, S. K.* $\mathscr{P}$-Differentiability on Carnot Groups in Different Topologies and Related Topics, *Trudy po Analizu i Geometrii* [Proceedings in Analysis and Geometry], Novosibirsk, Institute of Mathematics of SB RAS, 2000, pp. 603–670.
12. *Karmanova, M. B.* An Area Formula for Lipschitz Mappings of Carnot–Carathéodory Spaces, *Izvestiya: Mathematics*, 2014, vol. 78, no. 3, pp. 475–499. DOI: 10.1070/im2014v078n03abeh002695.

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МЕТРИЧЕСКИЕ ХАРАКТЕРИСТИКИ КЛАССОВ КОМПАКТНЫХ МНОЖЕСТВ НА ГРУППАХ КАРНО С СУБЛОРЕНЦЕВОЙ СТРУКТУРОЙ

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Аннотация. Мы рассматриваем классы отображений групп Карно, являющихся липшицевыми во внутреннем смысле и определенных на компактных подмножествах, и описываем метрические характеристики их образов при условии, что на области значений задана сублоренцева структура. Такая структура является субримановым обобщением геометрии Минковского. Одной из ее особенностей является неограниченность шаров, построенных относительно внутреннего расстояния. В сублоренцевой геометрии интерес представляет исследование пространственно-подобных поверхностей, пересечения которых с такими шарами ограничены. Если отображение определено на открытом множестве, то формулировка критерия пространственноподобия сводится к рассмотрению связной компоненты пересечения, содержащей центр шара, и анализу свойств матрицы субриманова дифференциала. Если же область определения отображения не является открытым множеством, то возникает вопрос, какие можно задать условия на отображение, гарантирующие ограниченность пересечения образа компактного множества с шаром во внутренней метрике. В данной статье этот вопрос решен: рассматривается та часть пересечения, которая параметризуется связной компонентой пересечения образа субриманова дифференциала и шара. Кроме того, с помощью таких локальных параметризаций введена функция множества, являющаяся аналогом меры Хаусдорфа, и показано, что она является мерой. В качестве приложения установлена сублоренцева формула площади.

Ключевые слова: группа Карно, липшицево отображение, компактное множество, сублоренцева структура, квазиаддитивная функция множества, формула площади.

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