

УДК 517.98

DOI 10.46698/y6929-3405-2251-o

ON COLLECTIVELY  $\sigma$ -LEVI SETS OF OPERATORS<sup>#</sup>

E. Yu. Emelyanov<sup>1</sup>

<sup>1</sup>Sobolev Institute of Mathematics,  
4 Ac. Koptyuga Ave., Novosibirsk 630090, Russia

E-mail: emelanov@math.nsc.ru

*Dedicated to Professor A. V. Abanin  
on occasion of his 70th birthday*

**Abstract.** The Levi operators are operator abstractions of the Levy property of Banach lattices. Such operators have been studied recently by several authors. The present paper deals with the collective properties of the Levi operators of several kinds:  $\sigma$ -Levi operators; quasi  $c$ - $\sigma$ -Levi operators; and quasi  $\sigma$ -Levi operators. A notion of collectively  $\sigma$ -Levi set generalizes the notion of a single  $\sigma$ -Levi operator to the families of operators. Working with families of sequences of elements of a vector lattice requires the notion of the collective order convergence. This notion that is introduced and studied in the present paper may have its own interest and further possible applications. Various relations of the collectively quasi  $\sigma$ -Levi sets to the collectively compact sets are investigated. The domination problem for the collectively quasi  $\sigma$ -Levi sets is studied. In this study a special notion of a set of operators dominated by another set of operators is used.

**Keywords:** vector lattice, normed lattice, collective order convergence, collectively  $\sigma$ -Levi set, collectively compact set.

**AMS Subject Classification:** 46A40, 46B42, 47L05.

**For citation:** Emelyanov, E. Yu. On Collectively  $\sigma$ -Levi Sets of Operators, *Vladikavkaz Math. J.*, 2025, vol. 27, no. 1, pp. 36–43. DOI: 10.46698/y6929-3405-2251-o.

## 1. Introduction

Several kinds of Levi operators were studied recently in [1–4] The present paper concerns collective properties of  $\sigma$ -Levi operators.

In what follows, vector spaces are real and operators are linear. The letters  $E$  and  $F$  stand for vector lattices, symbols  $L(E, F)$ ,  $L_{FR}(E, F)$ , and  $K(E, F)$  for the spaces of linear, finite rank, and compact operators from  $E$  to  $F$ ,  $B_X$  for the closed unit ball of  $X$ , and  $I_X$  for the identity operator in  $X$ . We write  $y_n \downarrow 0$ , whenever  $y_{n'} \leq y_n$  for all  $n' \geq n$  and  $\inf_E y_n = 0$ .

Throughout the paper, we say that a sequence  $(x_n)$  in  $E$  is order convergent to  $x \in E$  (briefly,  $x_n \xrightarrow{o} x$ ) if there exists a sequence  $(p_n)$  in  $E$ ,  $p_n \downarrow 0$  such that  $|x_n - x| \leq p_n$  holds for all  $n$ . A sequence  $(x_n)$  in  $E$  is order Cauchy if, for some  $p_n \downarrow 0$  in  $E$ ,  $|x_{n'} - x_{n''}| \leq p_n$

---

<sup>#</sup> The research was carried out within the framework of the state contract of the Sobolev Institute of Mathematics, project № FWNF-2022-0004.

© 2025 Emelyanov, E. Yu.

whenever  $n', n'' \geq n$ . A vector lattice  $E$  is said to be sequentially order complete whenever each order Cauchy sequence in  $E$  is order convergent.

The following definition is an adopted version of [1, Definition 1.1] and [3, Definition 1].

DEFINITION 1. An operator  $T$  from a normed lattice  $E$  to a vector lattice  $F$  is:

a)  $\sigma$ -Levi if, for every increasing bounded sequence  $(x_n)$  in  $E_+$ , there exists  $x \in E$  with  $Tx_n \xrightarrow{o} Tx$ . The set of such operators is denoted by  $L_{\text{Levi}}^\sigma(E, F)$ .

b) quasi-c- $\sigma$ -Levi if, for every increasing bounded sequence  $(x_n)$  in  $E_+$ , the sequence  $(Tx_n)$  is order convergent. The set of such operators is denoted by  $L_{\text{qcLevi}}^\sigma(E, F)$ .

c) quasi- $\sigma$ -Levi if, for every increasing bounded sequence  $(x_n)$  in  $E_+$ , the sequence  $(Tx_n)$  is order Cauchy. The set of such operators is denoted by  $L_{\text{qLevi}}^\sigma(E, F)$ .

Clearly,  $L_{\text{Levi}}^\sigma(E, F) \subseteq L_{\text{qcLevi}}^\sigma(E, F) \subseteq L_{\text{qLevi}}^\sigma(E, F)$ . The following example shows that both inclusions are proper in general (cf., [2, Example 1]).

EXAMPLE 1. First we show that the inclusion  $L_{\text{Levi}}^\sigma(E) \subseteq L_{\text{qcLevi}}^\sigma(E)$  can be proper. Define an operator  $T$  on  $E = C[0, 1] \oplus L_1[0, 1]$ , by  $T((\phi, \psi)) := (0, \phi)$  for  $\phi \in C[0, 1]$  and  $\psi \in L_1[0, 1]$ . Clearly,  $T \in L_{\text{qcLevi}}^\sigma(E)$ . Consider  $\phi_n \in C[0, 1]$  that equals to 1 on  $[0, \frac{1}{2} - \frac{1}{2^n}]$ , to 0 on  $[\frac{1}{2}, 1]$ , and is linear otherwise. Let  $f_n := (\phi_n, 0)$ . Then  $(f_n)$  is bounded and increasing in  $E_+$ , and  $Tf_n \xrightarrow{o} (0, g)$ , where  $g \in L_1[0, 1]$  is the indicator function of  $[0, \frac{1}{2}]$ . Since  $g \notin C[0, 1]$ , there is no such an  $f \in E$  that  $Tf = (0, g)$ , and hence  $T \notin L_{\text{Levi}}^\sigma(E)$ .

For the second inclusion, consider the Banach lattice  $c$  of convergent real sequences and denote elements of  $c$  by  $\sum_{n=1}^{\infty} a_n \cdot e_n$ , where  $e_n$  is the  $n$ -th unit vector of  $c$  and  $(a_n)$  converges in  $\mathbb{R}$ . Since each bounded increasing sequence in  $c_+$  is o-Cauchy,  $I_c \in L_{\text{qLevi}}^\sigma(c)$ . However, a bounded increasing sequence  $I_c f_n = f_n := \sum_{k=1}^n e_{2k-1}$  in  $c_+$  is not order convergent. Thus,  $I_c \notin L_{\text{qcLevi}}^\sigma(c)$ .

We shall use the following Lemma (cf., [2, Lemmas 1, 2]).

**Lemma 1.** Let  $E$  be a normed lattice and let  $F$  be a vector lattice. The following holds.

- i)  $L_{\text{qcLevi}}^\sigma(E, F)$  and  $L_{\text{qLevi}}^\sigma(E, F)$  are vector spaces.
- ii)  $L_{\text{FR}}(E, F) \subseteq L_{\text{Levi}}^\sigma(E, F)$ .
- iii) If  $F$  is a normed lattice then  $K_+(E, F) \subseteq L_{\text{qcLevi}}^\sigma(E, F)$ .

$\triangleleft$  i) It is trivial.

ii) Let  $T \in L_{\text{FR}}(E, F)$ , say  $T = \sum_{k=1}^n f_k \otimes y_k$  for  $y_1, \dots, y_n \in T(E)$  and  $f_1, \dots, f_n \in E'$ .

Denote

$$T_1 := \sum_{k=1}^n f_k^+ \otimes y_k, \quad T_2 := \sum_{k=1}^n f_k^- \otimes y_k.$$

Let  $(x_m)$  be an increasing bounded sequence in  $E_+$ . Then, for each  $k$ , the sequences  $f_k^+(x_m)$  and  $f_k^-(x_m)$  are increasing and bounded. Thus,  $f_k^+(x_m) \rightarrow a_k$  and  $f_k^-(x_m) \rightarrow b_k$  for some  $a_k, b_k \in \mathbb{R}_+$ . Since  $\dim(T(E)) < \infty$ ,

$$T_1 x_m \xrightarrow{o} \sum_{k=1}^n a_k y_k \in T(E), \quad T_2 x_m \xrightarrow{o} \sum_{k=1}^n b_k y_k \in T(E).$$

Therefore,

$$Tx_m = (T_1 x_m - T_2 x_m) \xrightarrow{o} \sum_{k=1}^n (a_k - b_k) y_k \in T(E).$$

Take an  $x \in E$ , such that  $Tx = \sum_{k=1}^n (a_k - b_k) y_k$ . Then  $Tx_m \xrightarrow{o} Tx$ . We conclude  $T \in L_{\text{Levi}}^\sigma(E, F)$ .

iii) Let  $T \in K_+(E, F)$  and let  $(x_m)$  be an increasing sequence in  $(B_E)_+$ . Then  $(Tx_m)$  has a subsequence  $(Tx_{m_j})$  satisfying  $\|Tx_{m_j} - y\| \rightarrow 0$  for some  $y \in F$ . Since  $Tx_m \uparrow$  then  $\|Tx_m - y\| \rightarrow 0$ . As each norm convergent increasing sequence converges in order to the same limit then  $Tx_m \xrightarrow{o} y$ , and consequently  $T \in L_{\text{qcLevi}}^\sigma(E, F)$ .  $\triangleright$

The next example strengthens Example 1 by showing that the inclusion  $L_{\text{Levi}}^\sigma(E) \cap K_+(E) \subseteq L_{\text{qcLevi}}^\sigma(E) \cap K_+(E)$  can be proper (cf., [2, Example 2]).

EXAMPLE 2. Let  $(\alpha_n)$  be a vanishing real sequence consisting of non-zero positive terms. Define an operator  $S$  from  $c$  to  $c_0$  by  $S(\sum_{n=1}^\infty a_n e_n) = \sum_{n=1}^\infty (\alpha_n a_n) e_n$ . Then  $S \in K_+(c, c_0)$ , and hence  $S \in L_{\text{qcLevi}}^\sigma(c, c_0)$  by Lemma 1. Take a bounded increasing sequence  $x_n = \sum_{k=1}^n e_{2k}$  in  $c_+$ . The sequence  $(Sx_n) = (\sum_{k=1}^n \alpha_{2k} e_{2k})$  converges in order to  $\sum_{k=1}^\infty \alpha_{2k} e_{2k} \in c_0$ , however there is no  $x \in c$  with  $Sx = \sum_{k=1}^\infty \alpha_{2k} e_{2k}$ . Indeed, would such  $x = \sum_{k=1}^\infty a_k e_k \in c$  with  $Sx = S(\sum_{k=1}^\infty a_k e_k) = \sum_{k=1}^\infty \alpha_{2k} e_{2k}$  exist, it must satisfies  $a_k = 1$  for even  $k$ -th and  $a_k = 0$  for odd  $k$ -th, which is absurd. Therefore,  $S \notin L_{\text{Levi}}^\sigma(c, c_0)$ .

The operator  $S$  is also a counter-example to [1, Proposition 3.5].

Now, define a sequence  $(S_i)$  of operators in  $L_{\text{FR}}(c, c_0)$  by  $S_i(\sum_{n=1}^\infty a_n e_n) = \sum_{n=1}^i (\alpha_n a_n) e_n$ . Trivially,  $S_i \xrightarrow{\|\cdot\|} S$ . By Lemma 1,  $S_i \in L_{\text{Levi}}^\sigma(c, c_0)$ . Since  $S \notin L_{\text{Levi}}^\sigma(c, c_0)$  then the set  $L_{\text{Levi}}^\sigma(c, c_0)$  is not closed under the operator norm.

It is worth noting that, generally,  $L_{\text{Levi}}^\sigma(E, F)$  need not to be a vector space [2, Example 8].

The present paper is organized as follows. Section 2 is devoted to elementary properties of collective order convergence of families of sequences. Section 3 is devoted to collectively  $\sigma$ -Levi sets of operators, their relations to collectively compact sets, and for the domination problem.

For unexplained terminology and notation we refer to [5–8].

## 2. Collective Order Convergence

Working with families of sequences of elements of a vector lattice requires a certain notion of “collective” order convergence. In what follows, we identify  $E$ -valued sequences and elements of the vector lattice  $E^{\mathbb{N}}$  equipped with the pointwise linear and lattice operations.

DEFINITION 2. Let  $\mathcal{A} \subseteq E^{\mathbb{N}}$ . We say that  $\mathcal{A}$  collective order converges to an indexed subset  $\{c_a\}_{a \in \mathcal{A}}$  of  $E$  (briefly,  $\mathcal{A} \xrightarrow{c-o} \{c_a\}_{a \in \mathcal{A}}$ ) whenever there exists a sequence  $(p_n)$  in  $E$ ,  $p_n \downarrow 0$  such that  $|a_n - c_a| \leq p_n$  holds for all  $n$  and all  $(a_n) \in \mathcal{A}$ . We call  $\mathcal{A}$  collective order-null if  $\mathcal{A} \xrightarrow{c-o} \{0\}_{a \in \mathcal{A}}$ .

In this section we give some elementary properties of collective order convergence which are used in Section 3. The following proposition is elementary and its proof is left to the reader.

**Proposition 1.** *Let  $\mathcal{A}$  and  $\mathcal{B}$  be nonempty collective order convergent subsets of  $E^{\mathbb{N}}$ , and let  $\alpha, \beta \in \mathbb{R}$ . The following sets are collective order convergent.*

- i)  $\mathcal{A} \cup \mathcal{B}$ .
- ii)  $\alpha\mathcal{A} + \beta\mathcal{B} := \{(\alpha a_n + \beta b_n)\}_{a \in \mathcal{A}; b \in \mathcal{B}}$ .
- iii)  $|\mathcal{A}| := \{(|a_n|)\}_{a \in \mathcal{A}}$ .
- iv) The convex hull  $\text{co}(\mathcal{A})$  of  $\mathcal{A}$  in  $E^{\mathbb{N}}$ .
- Moreover, v) If  $\mathcal{A} \xrightarrow{c-o} \{c_a\}_{a \in \mathcal{A}}$  and  $\mathcal{A} \xrightarrow{c-o} \{c'_a\}_{a \in \mathcal{A}}$  then  $c_a = c'_a$  for all  $a \in \mathcal{A}$ .
- vi)  $\mathcal{A} \xrightarrow{c-o} \{c_a\}_{a \in \mathcal{A}}$  iff  $\{(a_n - c_a)\}_{a \in \mathcal{A}} \xrightarrow{c-o} \{0\}_{a \in \mathcal{A}}$ .
- vii) A sequence  $(a_n)$  in  $E$  order converges iff the set  $\{(a_n)\}$  is collective order convergent.

Note that the passing to solid hull does not preserve collective order convergence for any nontrivial  $E$ . Indeed, let  $0 \neq x \in E$ . Then the set  $\mathcal{A} = \{(a_n) : a_n \equiv x\}$  of one

constantly  $x$  sequence is collective order convergent, yet its solid hull  $\text{sol}(\mathcal{A})$  is not, as  $\text{sol}(\mathcal{A})$  contains a sequence  $\left(\frac{1+(-1)^n}{2}x\right)$  that does not order converge. It should be clear that if  $E$  is an Archimedean vector lattice then, for each order convergent to zero sequence  $(a_n)$  in  $E$  possessing at least one non-zero term, the set  $\{(\lambda a_n) : \lambda \in \mathbb{R}\}$  is not collective order-null. Also, it is worth noting that the set  $\{(\delta_k^n) : k \in \mathbb{N}\}$  consisting of order-null real sequences is not collective order-null.

The next theorem extends items ii) and iv) of Proposition 1 to the Banach lattice setting as follows.

**Theorem 1.** *Let  $E$  be a Banach lattice, let  $(p_{i,n})_n$  be sequences in  $B_E$  satisfying  $p_{i,n} \downarrow 0$  for each  $i \in \mathbb{N}$ , and let  $\mathcal{A}_i$  be nonempty subsets of  $E^{\mathbb{N}}$ , such that  $|a_{i,n}| \leq p_{i,n}$  holds for all  $i, n \in \mathbb{N}$ , and  $(a_{i,n})_n \in \mathcal{A}_i$ . Then, the set*

$$\sum_{i=1}^{\infty} \alpha_i \mathcal{A}_i = \left\{ \left( \sum_{i=1}^{\infty} \alpha_i a_{i,n} \right) : (a_{i,n})_n \in \mathcal{A}_i, \sum_{i=1}^{\infty} |\alpha_i| \leq 1 \right\}$$

is collective order-null. In particular, for every  $M > 0$  and  $\mathcal{A} \xrightarrow{c-o} \{0\}_{a \in \mathcal{A}}$  in  $E^{\mathbb{N}}$ , the set

$$\left\{ \left( \sum_{i=1}^{\infty} \alpha_i a_{i,n} \right) : (a_{i,n})_n \in \mathcal{A}, \sum_{i=1}^{\infty} |\alpha_i| \leq M \right\}$$

is collective order-null.

◁ Passing to the norm-limit as  $m \rightarrow \infty$  in the following inequality

$$\left| \sum_{i=1}^m \alpha_i a_{i,n} \right| \leq \sum_{i=1}^m |\alpha_i| |a_{i,n}| \leq \sum_{i=1}^m |\alpha_i| p_{i,n} \leq \sum_{i=1}^{\infty} |\alpha_i| p_{i,n},$$

where  $(a_{i,n})_n \in \mathcal{A}_i$ , we obtain  $\left| \sum_{i=1}^{\infty} \alpha_i a_{i,n} \right| \leq p_n := \sum_{i=1}^{\infty} |\alpha_i| p_{i,n}$  for all  $n$ . Clearly,  $(p_n)$  is decreasing. It remains to proof  $p_n \downarrow 0$ . Suppose in contrary  $0 < a \leq p_n$  for all  $n$ . Fix an arbitrary  $m \in \mathbb{N}$ . Since  $0 < a \leq \sum_{i=1}^m |\alpha_i| p_{i,n} + \sum_{i=m+1}^{\infty} |\alpha_i| p_{i,n}$  for all  $n$ , and since  $\inf_{n \in \mathbb{N}} \sum_{i=1}^m |\alpha_i| p_{i,n} = 0$ , we obtain that  $0 < a \leq \sum_{i=m+1}^{\infty} |\alpha_i| p_{i,n}$  for all  $m, n \in \mathbb{N}$ . Therefore,

$$0 < \|a\| \leq \limsup_{m \rightarrow \infty} \left\| \sum_{i=m+1}^{\infty} |\alpha_i| p_{i,n} \right\| \leq \lim_{m \rightarrow \infty} \sum_{i=m+1}^{\infty} |\alpha_i| = 0,$$

which is absurd.

The rest of proof follows from the previous part by taking  $\mathcal{A}_i = \mathcal{A}$  for all  $i \in \mathbb{N}$ . ▷

We finish this section with the following notion of collective order Cauchy set of sequences.

**DEFINITION 3.** A set  $\mathcal{A} \subseteq E^{\mathbb{N}}$  is collective order Cauchy if, for some  $p_n \downarrow 0$  in  $E$ ,  $|a_{n'} - a_{n''}| \leq p_n$  holds for all  $a \in \mathcal{A}$  whenever  $n', n'' \geq n$ . A vector lattice  $E$  is sequentially collective order complete if each collective order Cauchy subset of  $E^{\mathbb{N}}$  is collective order convergent.

The next elementary proposition shows that the sequential collective order completeness agrees with sequential order completeness.

**Proposition 2.** *Let  $E$  be a vector lattice. The following conditions are equivalent.*

i) *If a sequence  $(x_n)$  in  $E$  satisfies  $|x_{n'} - x_{n''}| \leq p_n$ , whenever  $n', n'' \geq n$  for some  $p_n \downarrow 0$  in  $E$ , then there exists  $x \in E$  with  $|x_n - x| \leq p_n$  for all  $n$ .*

ii) If a subset  $\mathcal{A}$  of  $E^{\mathbb{N}}$  satisfies  $|a_{n'} - a_{n''}| \leq p_n$  for all  $a \in \mathcal{A}$  and some  $p_n \downarrow 0$  in  $E$ , whenever  $n', n'' \geq n$ , then there exists an indexed subset  $\{c_a\}_{a \in \mathcal{A}}$  of  $E$ , such that  $|a_n - c_a| \leq p_n$  holds for all  $n$  and all  $(a_n) \in \mathcal{A}$ .

iii)  $E$  is sequentially collective order complete.

iv)  $E$  is sequentially order complete.

$\triangleleft$  Implications i)  $\implies$  ii)  $\implies$  iii)  $\implies$  iv) are trivial.

iv)  $\implies$  i) Let  $|x_{n'} - x_{n''}| \leq p_n$  for all  $n', n'' \geq n$  and some  $p_n \downarrow 0$  in  $E$ . Since  $E$  is sequentially order complete,  $x_n \xrightarrow{o} x$  for some  $x \in E$ . Sending  $n'' \rightarrow \infty$  and passing to the order limit in the inequality  $|x_n - x_{n''}| \leq p_n$ , where  $n'' \geq n$ , we obtain  $|x_n - x| \leq p_n$  for all  $n$ .  $\triangleright$

### 3. Collectively $\sigma$ -Levi Sets of Operators

Recall that a set  $A$  of operators between normed spaces  $X$  and  $Y$  is collectively compact whenever  $\bigcup_{T \in A} T(B_X)$  is relatively compact in  $Y$  [6]. This section is devoted to collectively  $\sigma$ -Levi sets of operators, their relation to collectively compact sets, and the domination problem for collectively  $\sigma$ -Levi sets. We begin with the following collective version of Definition 1.

DEFINITION 4. Let  $E$  be a normed lattice,  $F$  a vector lattice, and  $A \subseteq L(E, F)$ . We say that  $A$  is:

a) a collectively  $\sigma$ -Levi set if, for every increasing bounded  $(x_n)$  in  $E_+$ , there exists an indexed subset  $\{x_T\}_{T \in A}$  of  $E$  satisfying  $\{(Tx_n) : T \in A\} \xrightarrow{c-o} \{Tx_T\}_{T \in A}$ .

b) a collectively quasi-c- $\sigma$ -Levi set if, for every increasing bounded  $(x_n)$  in  $E_+$ , there exists an indexed subset  $\{y_T\}_{T \in A}$  of  $F$  satisfying  $\{(Tx_n) : T \in A\} \xrightarrow{c-o} \{y_T\}_{T \in A}$ .

c) a collectively quasi- $\sigma$ -Levi set if, for every increasing bounded  $(x_n)$  in  $E_+$ , the set  $\{(Tx_n) : T \in A\} \subseteq F^{\mathbb{N}}$  is collective order Cauchy.

Obviously,  $T$  lies in  $L_{\text{Levi}}^{\sigma}(E, F)$  ( $L_{\text{qcLevi}}^{\sigma}(E, F)$ ,  $L_{\text{qLevi}}^{\sigma}(E, F)$ ) iff the set  $\{T\}$  is a collectively  $\sigma$ -Levi (resp., collectively quasi-c- $\sigma$ -Levi, collectively quasi- $\sigma$ -Levi) subset of  $L(E, F)$ .

We continue with the question on which properties of  $\sigma$ -Levi, quasi-c- $\sigma$ -Levi, and quasi- $\sigma$ -Levi operators mentioned in Lemma 1 have collective versions. The properties described in Lemma 1 i) have the following extension.

**Proposition 3.** Let  $E$  be a normed lattice,  $F$  a vector lattice, and  $A, B \subseteq L(E, F)$ . The following holds.

i) If  $A$  and  $B$  are both collectively quasi-c- $\sigma$ -Levi then the set  $\{\alpha T + \beta S : |\alpha| + |\beta| \leq 1, T \in A, S \in B\}$  is also collectively quasi-c- $\sigma$ -Levi.

ii) If  $A$  and  $B$  are both collectively quasi- $\sigma$ -Levi then the set  $\{\alpha T + \beta S : |\alpha| + |\beta| \leq 1, T \in A, S \in B\}$  is also collectively quasi- $\sigma$ -Levi.

$\triangleleft$  i) By the assumption, there exist sequences  $p_n \downarrow 0, q_n \downarrow 0$  in  $F$ , and indexed subsets  $\{y_T\}_{T \in A}, \{z_S\}_{S \in B}$  of  $F$  satisfying  $|Tx_n - y_T| \leq p_n, |Sx_n - z_S| \leq q_n$  for all  $T \in A, S \in B$ , and  $n \in \mathbb{N}$ . The result follows from

$$|(\alpha T + \beta S)x_n - (\alpha y_T + \beta z_S)| \leq |\alpha|p_n + |\beta|q_n \leq (p_n + q_n) \downarrow 0.$$

ii) Let sequences  $(p_n), (q_n)$  in  $F$  satisfy  $|Tx_{n'} - Tx_{n''}| \leq p_n \downarrow 0$  and  $|Sx_{n'} - Sx_{n''}| \leq q_n \downarrow 0$  for all  $T \in A, S \in B$ , and  $n', n'' \geq n$ . The result follows from

$$|(\alpha T + \beta S)x_{n'} - (\alpha T + \beta S)x_{n''}| \leq (p_n + q_n) \downarrow 0,$$

for  $n', n'' \geq n$ .  $\triangleright$

The items ii) and iii) of Lemma 1 have no reasonable collective extension. To see this, define norm-one functionals  $T_k$  on  $c_0$  by  $T_k a = a_k$ . Thus,  $T_k \in L_{FR}(c_0, \mathbb{R})$ , yet the set  $\{T_k\}_{k \in \mathbb{N}}$  is not even collectively quasi- $\sigma$ -Levi. Indeed, for the increasing bounded sequence  $x_n = \sum_{m=1}^n e_m$  in  $c_0$ , there is no sequence  $p_n \downarrow 0$  in  $\mathbb{R}$  with  $|T_k x_{n'} - T_k x_{n''}| \leq p_n$  for all  $k$  and  $n', n'' \geq n$ , since  $|T_{n+1} x_n - T_{n+1} x_{n+1}| = 1$  for every  $n$ . Moreover,  $\{T_k\}_{k \in \mathbb{N}}$  is a collectively compact subset of  $L_+(c_0, \mathbb{R})$  that is not collectively quasi- $\sigma$ -Levi.

Now, we apply Theorem 1 for strengthening Proposition 3 in the Banach lattice setting as follows.

**Theorem 2.** *Let  $E$  be a normed lattice,  $F$  a Banach lattice, and let  $A$  be a bounded collectively quasi- $c$ - $\sigma$ -Levi subset of  $L(E, F)$ . Then the set*

$$\left\{ \left( \sum_{i=1}^{\infty} \alpha_i T_i \right) : T_i \in A, \sum_{i=1}^{\infty} |\alpha_i| \leq 1 \right\}$$

is collectively quasi- $c$ - $\sigma$ -Levi, where  $\sum_{i=1}^{\infty} \alpha_i T_i$  is the limit of partial sums  $\sum_{i=1}^n \alpha_i T_i$  in the operator norm.

$\triangleleft$  Let  $(x_n) \uparrow$  in  $(B_E)_+$ . Then  $\{(Tx_n)\}_{T \in A} \xrightarrow{c-o} \{y_T\}_{T \in A}$  for some subset  $\{y_T\}_{T \in A}$  of  $F$ . Proposition 1 vi) gives  $\{(Tx_n - y_T)\}_{T \in A} \xrightarrow{c-o} \{0\}_{T \in A}$ . By Theorem 1,

$$\left\{ \left( \sum_{i=1}^{\infty} \alpha_i (T_i x_n - y_{T_i}) \right) \right\}_{\substack{T_i \in A, \\ \sum_{i=1}^{\infty} |\alpha_i| \leq 1}} \xrightarrow{c-o} \{0\}_{\substack{T_i \in A, \\ \sum_{i=1}^{\infty} |\alpha_i| \leq 1}}.$$

So, there exists a sequence  $p_n \downarrow 0$  in  $F$  satisfying

$$\left| \left( \sum_{i=1}^{\infty} \alpha_i T_i \right) x_n - \sum_{i=1}^{\infty} \alpha_i y_{T_i} \right| = \left| \sum_{i=1}^{\infty} \alpha_i (T_i x_n - y_{T_i}) \right| \leq p_n$$

for all  $n$ ,  $T_i \in A$ , and all  $\alpha_i$  with  $\sum_{i=1}^{\infty} |\alpha_i| \leq 1$ , where the series  $\sum_{i=1}^{\infty} \alpha_i T_i$  converges in the operator norm due to boundedness of  $A$ . The proof is complete.  $\triangleright$

Since every Dedekind  $\sigma$ -complete vector lattice is sequentially order complete, the next corollary follows from Proposition 2 and Theorem 2.

**Corollary 1.** *Let  $E$  be a normed lattice,  $F$  be a Dedekind  $\sigma$ -complete Banach lattice, and  $A$  be a bounded collectively quasi- $\sigma$ -Levi subset of  $L(E, F)$ . Then the set*

$$\left\{ \left( \sum_{i=1}^{\infty} \alpha_i T_i \right) : T_i \in A, \sum_{i=1}^{\infty} |\alpha_i| \leq 1 \right\}$$

is collectively quasi- $c$ - $\sigma$ -Levi.

Now, we discuss of the ‘‘collective’’ domination problem for Levi sets of operators. First, recall some already known related results for Levi operators.

The quasi- $\sigma$ -Levi operators do satisfy the domination property (cf. [1, Theorem 2.7], [3, Theorem 3]). We do not know where quasi- $c$ - $\sigma$ -Levi operators satisfy the domination property. In general,  $\sigma$ -Levi operators do not satisfy the domination property (cf. [2, Example 7]).

EXAMPLE 3. Define operators  $S, T \in L(c)$  by

$$S \left( \sum_{n=1}^{\infty} a_n e_n \right) = \sum_{n=1}^{\infty} \frac{a_n}{2^n} e_n; T \left( \sum_{n=1}^{\infty} a_n e_n \right) = \sum_{n=1}^{\infty} \left( \sum_{k=1}^{\infty} \frac{a_k}{2^k} \right) e_n.$$

Then  $0 \leq S \leq T$ . Operator  $T$  has rank one, and hence  $T$  is  $\sigma$ -Levi by Lemma 1 *ii*). However,  $S \notin L_{\text{Levi}}^{\sigma}(c)$  due to Example 2.

We use the following “collective” notion of domination for sets of operators.

DEFINITION 5. Let  $A, B \subseteq L_+(E, F)$ . Then  $A$  is dominated by  $B$  if, for each  $S \in A$ , there exists  $T \in B$  with  $S \leq T$ .

We conclude the paper with the following “collective” partial generalization of [1, Theorem 2.7] in the class of quasi- $\sigma$ -Levi operators.

**Theorem 3.** *Let  $E$  be a normed lattice,  $F$  be a vector lattice, and  $A, C \subseteq L_+(E, F)$  be such that  $A$  is dominated by  $C$ . If  $C$  is collectively quasi- $\sigma$ -Levi then  $A$  is also collectively quasi- $\sigma$ -Levi.*

◁ Let  $(x_n)$  be an increasing sequence in  $(B_E)_+$ . By the assumption,  $C$  is collectively quasi- $\sigma$ -Levi, and hence the set  $\{(Tx_n) : T \in C\} \subseteq F^{\mathbb{N}}$  is collective order Cauchy. By Definition 3, for some  $p_n \downarrow 0$  in  $F$ ,  $|Tx_{n'} - Tx_{n''}| \leq p_n$  holds for all  $T \in C$  whenever  $n', n'' \geq n$ .

Let  $S \in A$ . Then  $0 \leq S \leq T_S$  for some  $T_S \in C$ . Since  $|T_S x_{n'} - T_S x_{n''}| \leq p_n$  for  $n', n'' \geq n$ ,

$$\begin{aligned} |Sx_{n'} - Sx_{n''}| &\leq |Sx_{n'} - Sx_n| + |Sx_{n''} - Sx_n| \\ &= S(x_{n'} - x_n) + S(x_{n''} - x_n) \leq T_S(x_{n'} - x_n) + T_S(x_{n''} - x_n) \\ &= |T_S x_{n'} - T_S x_n| + |T_S x_{n''} - T_S x_n| \leq 2p_n \end{aligned}$$

for all  $n', n'' \geq n$ . Because  $S \in A$  is arbitrary and  $2p_n \downarrow 0$ , we conclude that  $A$  is collectively quasi- $\sigma$ -Levi. ▷

## References

1. Alpay, S., Emelyanov, E. and Gorokhova, S.  $\sigma$ -Continuous, Lebesgue, KB, and Levi Operators Between Vector Lattices and Topological Vector Spaces, *Results in Mathematics*, 2022, vol. 77, no. 3, article no. 117, pp. 1–25. DOI: 10.1007/s00025-022-01650-3.
2. Emelyanov, E. On KB and Levi Operators in Banach Lattices, arXiv:2312.05685v2 [math.FA], 2023. DOI: 10.48550/arXiv.2312.05685.
3. Gorokhova, S. G. and Emelyanov, E. Y. On Operators Dominated by Kantorovich–Banach Operators and Levy Operators in Locally Solid Lattices, *Vladikavkaz Math. J.*, 2022, vol. 24, no. 3, pp. 55–61 (in Russian). DOI: 10.46698/f5525-0005-3031-h.
4. Zhang, F. and Chen, Z. Some Results of  $(\sigma)$ -Levi Operators in Banach Lattices, *Positivity*, 2022, vol. 26, article no. 49. DOI: 10.1007/s11117-022-00903-3.
5. Aliprantis, C. D. and Burkinshaw, O. *Locally Solid Riesz Spaces with Applications to Economics: Second Edition, Mathematical Surveys and Monographs*, vol. 105, Providence, RI, American Mathematical Society, 2003.
6. Anselone, P. M. and Palmer, T. W. Collectively Compact Sets of Linear Operators, *Pacific Journal of Mathematics*, 1968, vol. 25, no. 3, pp. 417–422.
7. Kusraev, A. G. *Dominated Operators*, Dordrecht, Kluwer, 2000.
8. Meyer-Nieberg, P. *Banach Lattices*, Berlin, Springer-Verlag, Universitext, 1991.

Received May 22, 2024

EDUARD YU. EMELYANOV  
Sobolev Institute of Mathematics,  
4 Ac. Koptyuga Ave., Novosibirsk 630090, Russia,  
Leading Researcher  
E-mail: emelanov@math.nsc.ru  
<https://orcid.org/0000-0002-8828-0398>

СОВМЕСТНО  $\sigma$ -ЛЕВИ МНОЖЕСТВА ОПЕРАТОРОВЕмельянов Э. Ю.<sup>1</sup><sup>1</sup> Институт математики им. С. Л. Соболева СО РАН,  
Россия, 630090, Новосибирск, пр-т Академика Коптюга, 4  
E-mail: emelanov@math.nsc.ru

**Аннотация.** Операторы Леви являются операторной абстракцией соответствующего свойства банаховых решеток, известного как свойство Леви. Операторы такого рода в последнее время стали объектом пристального внимания нескольких авторов. В настоящей статье нами рассмотрены так называемые совместные свойства операторов Леви некоторых типов, а именно  $\sigma$ -Леви операторы, квази  $s$ - $\sigma$ -Леви операторы, а также квази  $\sigma$ -Леви операторы. Понятие совместно  $\sigma$ -Леви множества операторов расширяет понятие одного  $\sigma$ -Леви оператора на целое семейство таких операторов. При работе с семействами последовательностей, составленных из элементов векторной решетки, нам требуется понятие совместной порядковой сходимости. Это понятие, которое мы вводим и рассматриваем в данной статье, может представлять самостоятельный интерес и, возможно, даже имеет некоторые новые приложения в теории векторных решеток и теории операторов. В работе также исследуются вопросы существования различных взаимосвязей, которые возникают между множествами квази  $\sigma$ -Леви операторов и компактных операторов. Изучается вопрос мажорирования для совместно квази  $\sigma$ -Леви операторных множеств. В рамках этого исследования нами использовано понятие множества операторов, мажорируемого некоторым другим множеством операторов.

**Ключевые слова:** векторная решетка, нормированная решетка, совместная порядковая сходимость, совместно  $\sigma$ -Леви множество, совместно компактное множество.

**AMS Subject Classification:** 46A40, 46B42, 47L05.

**Образец цитирования:** Emelyanov E. Yu. On Collectively  $\sigma$ -Levi Sets of Operators // Владикавк. мат. журн.—2025.—Т. 27, № 1.—С. 36–43 (in English). DOI: 10.46698/y6929-3405-2251-o.