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ON A NEW CLASS OF MEROMORPHIC FUNCTIONS ASSOCIATED WITH MITTAG-LEFFLER FUNCTION

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Abstract. The Mittag-Leffler function arises naturally in solving differential and integral equations of fractional order and especially in the study of fractional generalization of kinetic equation, random walks, Lévy flights, super-diffusive transport and in the study of complex systems. In the present investigation, the authors define a new class $\mathfrak{M}^{\tau;\kappa}_{\varsigma,\varrho}(\vartheta,\wp)$ of meromorphic functions defined in the punctured unit disk $\Delta^* := \{z \in \mathbb{C} : 0 < |z| < 1\}$ based on Mittag-Leffler. We discuss extensively its characteristic properties like coefficient inequalities, growth and distortion inequalities, as well as closure results for $f \in \mathfrak{M}^{\tau;\kappa}_{\varsigma,\varrho}(\vartheta,\wp)$. Properties of a certain integral operator and its inverse defined on the new class $\mathfrak{M}^{\tau;\kappa}_{\varsigma,\varrho}(\vartheta,\wp)$ are also discussed. Coefficient inequalities, growth and distortion inequalities, as well as closure results are obtained. We also establish some results concerning neighborhoods and the partial sums of meromorphic functions in this new class. We also state some new subclasses and their characteristic properties by specializing the parameters which are new and not studied before in association with Mittag-Leffler functions.

Keywords: meromorphic functions, starlike function, convolution, positive coefficients, coefficient inequalities, integral operator, Mittag-Leffler function, Hilbert space operator.

AMS Subject Classification: 30C45, 30C50.

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1. Introduction

Let Σ denote the class of normalized meromorphic functions f of the form

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n, \tag{1.1}$$

defined on the unit disk

$$\Delta = \{z \in \mathbb{C} : 0 < |z| < 1\}$$

and which are analytic except for a set of poles of finite order on $\mathbf{U} = \{z \in \mathbb{C} : |z| < 1\}$. Denoted by Σ_P and be of the form

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n, \quad a_n \geqslant 0. \tag{1.2}$$

The Hadamard product or convolution of two functions $f(z)$ given by (1.2) and

$$g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} g_n z^n \tag{1.3}$$

is defined by

$$(f * g)(z) := \frac{1}{z} + \sum_{n=1}^{\infty} a_n g_n z^n.$$

The function $f \in \Sigma$ with $f(0) \neq 0$ is called meromorphic starlike of order ϑ $(0 \leqslant \vartheta < 1)$, if

$$\operatorname{Re}\left(-\frac{zf'(z)}{f(z)}\right) > \vartheta \quad (z \in \Delta). \tag{1.4}$$

The function $f \in \Sigma$ with $f'(0) \neq 0$ is called meromorphic convex of order ϑ $(0 \leqslant \vartheta < 1)$ if

$$\operatorname{Re}\left(-\frac{(zf'(z))'}{f'(z)}\right) > \vartheta \quad (z \in \Delta). \tag{1.5}$$

The class of all such functions are denoted by $\Sigma_K(\vartheta)$. Further, we denote $\Sigma_P^*(\vartheta) = \Sigma^*(\vartheta) \cap \Sigma_P$ and $\Sigma_K^*(\vartheta) = \Sigma_K(\vartheta) \cap \Sigma_P$.

Lemma 1 [1]**.** *Suppose that $\vartheta \in [0,1)$, $r \in (0,1]$ and the function f is of the form*

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n, \quad 0 < |z| < r, \tag{1.6}$$

with $b_n \geqslant 0$. Then the condition

$$\operatorname{Re}\left(-\frac{zf'(z)}{f(z)}\right) > \vartheta \quad \text{for } |z| < r \tag{1.7}$$

is equivalent to the condition

$$\sum_{n=1}^{\infty} (n+\vartheta) b_n r^{n+1} \leqslant 1 - \vartheta. \tag{1.8}$$

The condition (1.4) and the above lemma with $r = 1$ give the following corollary.

Corollary 1 [1]**.** *Let $f \in \Sigma_P$ be given by (1.2). Then $f \in \Sigma_P^*(\vartheta)$ if and only if*

$$\sum_{n=1}^{\infty} (n+\vartheta) a_n \leqslant 1 - \vartheta. \tag{1.9}$$

Various subclasses of Σ have been studied rather extensively by Clunie [2], Nehari and Netanyahu [3], Pommerenke [4, 5], Royster [6], and others (cf., e.g., Bajpai [7], Mogra et al. [8], Uralegaddi and Ganigi [9], Cho et al. [10], Aouf [11], and Uralegaddi and Somanatha [12]); see also Duren [13, pp. 29 and 137], and Srivastava and Owa ([14, pp. 86 and 429], also see [11]).

Complex analysis (complex function theory) initiated in the 18th century and has since become one of the important topics in mathematics. Because of its effective applicability to a wide range of concepts and problems, this domain has significantly wedged a wide range of research areas, including engineering, physics, and mathematics. Researchers exposed some unexpected connections between ostensibly disparate study fields. Mittag-Leffler function (M-LF) research is an unusual and fascinating combination of geometry and complex analysis that deals with the structure of analytic functions in the complex domain and other domains related to sciences and engineering, has been a topic that has inspired several researchers. It was first proposed in by Mittag-Leffler [15] function ascends naturally in the solution of fractional order differential and integral equations, and exclusively in the studies of fractional generalizing of kinetic equation, random walks, Lévy flights, super-diffusive transport and in the study of complex systems. Let $\mathbf{E}_\varsigma$ be the function defined by

$$\mathbf{E}_\varsigma(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\varsigma n+1)}, \quad z \in \mathbb{C},\ \varsigma \in \mathbb{C} \quad \text{with } \operatorname{Re}\varsigma > 0,$$

that was introduced by Mittag-Leffler [15] and commonly known as the *Mittag-Leffler function*. Wiman [16] defined a more general function $\mathbf{E}_{\varsigma,\varrho}$ generalizing the $\mathbf{E}_\varsigma$ Mittag-Leffler function, that is

$$\mathbf{E}_{\varsigma,\varrho}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\varsigma n+\varrho)}, \quad z \in \mathbb{C},\ \varsigma,\varrho \in \mathbb{C}, \quad \text{with } \operatorname{Re}\varsigma > 0,\ \operatorname{Re}\varrho > 0.$$

When $\varrho = 1$, it is abbreviated as $\mathbf{E}_\varsigma(z) = \mathbf{E}_{\varsigma,1}(z)$. Observe that the function $\mathbf{E}_{\varsigma,\varrho}$ contains many well-known functions as its special case, for example,

$$\mathbf{E}_{1,1}(z) = e^z, \quad \mathbf{E}_{1,2}(z) = \frac{e^z-1}{z}, \quad \mathbf{E}_{2,1}\left(z^2\right) = \cosh z,$$
$$\mathbf{E}_{2,1}\left(-z^2\right) = \cos z, \quad \mathbf{E}_{2,2}\left(z^2\right) = \frac{\sinh z}{z}, \quad \mathbf{E}_{2,2}\left(-z^2\right) = \frac{\sin z}{z},$$
$$\mathbf{E}_4(z) = \frac{1}{2}\left(\cos z^{\frac{1}{4}} + \cosh z^{\frac{1}{4}}\right), \quad \mathbf{E}_3(z) = \frac{1}{2}\left[e^{z^{\frac{1}{3}}} + 2e^{-\frac{1}{2}z^{\frac{1}{3}}}\cos\left(\frac{\sqrt{3}}{2}z^{\frac{1}{3}}\right)\right].$$

Numerous properties of Mittag-Leffler function and generalized Mittag-Leffler function can be originated, for example in [17–22]. We note that the above generalized Mittag-Leffler function $\mathbf{E}_{\varsigma,\varrho}$ does not belongs to the family $\mathscr{A}$, where $\mathscr{A}$ represents the class of functions whose members are of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbf{U}, \tag{1.10}$$

which are analytic in the open unit disk Δ and normalized by the conditions $f(0) = f'(0) - 1 = 0$. Let $\mathscr{S}$ be the subclass of $\mathscr{A}$ whose members are univalent in Δ. Thus, it is expected to define the following normalization of Mittag-Leffler function as below, due to Bansal and Prajapat [18]:

$$E_{\varsigma,\varrho}(z) := z\Gamma(\varrho)\,\mathbf{E}_{\varsigma,\varrho}(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(\varrho)}{\Gamma(\varsigma(n-1)+\varrho)} z^n, \tag{1.11}$$

that holds for the parameters $\varsigma,\varrho \in \mathbb{C}$ with $\operatorname{Re}\varsigma > 0$, $\operatorname{Re}\varrho > 0$, and $z \in \mathbb{C}$.

Moreover, Srivastava and Tomovski [23] introduced the generalized the Mittag-Leffler function, $E^{\tau,\kappa}_{\varsigma,\varrho}(z)(z \in \mathbb{C})$ in the form

$$E^{\tau,\kappa}_{\varsigma,\varrho}(z) = \sum_{n=0}^{\infty} \frac{(\tau)_{n\kappa}}{\Gamma(\varsigma n+\varrho)\ n!} z^n,$$

($\varrho, \varsigma, \tau \in \mathbb{C}$; $\mathrm{Re}(\varsigma) > \max\{0, \mathrm{Re}(\kappa) - 1\}$; $\mathrm{Re}(\kappa) > 0$) and proved that it is an entire function in the complex z-plane, where

$$(\tau)_\theta = \frac{\Gamma(\tau+\theta)}{\Gamma(\tau)} \begin{cases} \tau(\tau+1)\cdots(\tau+\theta-1), & \theta \neq 0; \\ 1, & \theta = 0 \end{cases}$$

is well known Pochammer symbol. Lately, Aouf and Mostafa [24] defined

$$\mathbf{M}^{\tau,\kappa}_{\varsigma,\varrho}(z) = \Gamma(\varrho)\left(z^{-1}E^{\tau,\kappa}_{\varsigma,\varrho}(z) - \frac{\Gamma(\tau+\kappa)}{\Gamma(\tau)\Gamma(\varsigma+\varrho)}\right)$$

with ($\varrho, \tau \in \mathbb{C}$; $\mathrm{Re}(\varsigma) > \max\{0, \mathrm{Re}(\kappa) - 1\}$; $\mathrm{Re}(\kappa) > 0$; $\mathrm{Re}(\varsigma) = 0$ when $\mathrm{Re}(\kappa) = 1$ with $\varrho \neq 0$) and introduced a new linear operator for $f \in \Sigma$ and discussed differential inequalities for meromorphic univalent functions. Now we define a new linear operator $\mathscr{I}^{\tau,\kappa,m}_{\varsigma,\varrho} : \Sigma_P \to \Sigma_P$ by

$$\begin{aligned} \mathscr{I}^{\tau,\kappa,m}_{\varsigma,\varrho} f(z) &= \mathbf{M}^{\tau,\kappa}_{\varsigma,\varrho}(z) * f(z), \\ \mathscr{I}^{\tau,\kappa,m}_{\varsigma,\varrho} f(z) &= \frac{1}{z} + \sum_{n=1}^{\infty} \frac{\Gamma(\tau+(n+1)\kappa)\Gamma(\varrho)}{\Gamma(\tau)\Gamma((n+1)\varsigma+\varrho)\ (n)!} a_n z^n, \quad z \in \Delta, \end{aligned}$$

where the symbol $(*)$ denotes the Hadamard product (or convolution). We define a new operator $\mathscr{I}^{\tau,\kappa,m}_{\varsigma,\varrho} : \Sigma \to \Sigma$ in terms of Hadamard product, as follows:

$$\begin{aligned} \mathscr{I}^{\tau,\kappa,0}_{\varsigma,\varrho} &= \mathscr{I}^{\tau,\kappa,m}_{\varsigma,\varrho} f(z), \\ \mathscr{I}^{\tau,\kappa,1}_{\varsigma,\varrho} &= (1-\ell)\mathscr{I}^{\tau,\kappa,m}_{\varsigma,\varrho} f(z) + \ell(\mathscr{I}^{\tau,\kappa,m}_{\varsigma,\varrho} f(z))', \\ &\vdots \\ \mathscr{I}^{\tau,\kappa,m}_{\varsigma,\varrho} &= \mathscr{I}^{\tau,\kappa,1}_{\varsigma,\varrho}\left(\mathscr{I}^{\tau,\kappa,m-1}_{\varsigma,\varrho}(\mathscr{I}^{\tau,\kappa,m}_{\varsigma,\varrho} f(z))\right). \end{aligned}$$

Thus,

$$\mathscr{I}^{\tau,\kappa,m}_{\varsigma,\varrho} f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} (1+(n-1)\ell)^m \ \frac{\Gamma(\tau+(n+1)\kappa)\Gamma(\varrho)}{\Gamma(\tau)\Gamma((n+1)\varsigma+\varrho)\ (n)!} a_n z^n, \quad z \in \Delta. \tag{1.12}$$

Shortly, we let

$$\mathscr{I}^{\tau,\kappa,m}_{\varsigma,\varrho} f(z) := \frac{1}{z} + \sum_{n=1}^{\infty} \mho_n a_n z^n, \tag{1.13}$$

where

$$\mho_n = \frac{\Gamma(\tau+(n+1)\kappa)\Gamma(\varrho)}{\Gamma(\tau)\Gamma((n+1)\varsigma+\varrho)\ (n)!} \left(1+(n-1)\ell\right)^m, \tag{1.14}$$

$$\mho_1 = \frac{\Gamma(\tau+2\kappa)\Gamma(\varrho)}{\Gamma(\tau)\Gamma(2\varsigma+\varrho)}. \tag{1.15}$$

One can see that $\mathscr{I}^{1,1,0}_{0,\varrho} f(z) = zf'(z) + f(z) + z^{-1}$.

Let $\mathbb{H}$ be a complex Hilbert space and let $\mathscr{L}(\mathbb{H})$ denote the algebra of all bounded linear operators on $\mathbb{H}$. For a complex-valued function $f$analytic in a domain $\mathbb{E}$ of the complex z-plane containing the spectrum $\sigma(\mathbb{P})$ of the bounded linear operator $\mathbb{P}$, let $f(\mathbb{P})$ denote the operator on $\mathbb{H}$ defined by [25, p. 568]

$$f(\mathbb{P}) = \frac{1}{2\pi i}\int_{\mathscr{C}} (z\mathbb{I} - \mathbb{P})^{-1} f(z)dz, \tag{1.16}$$

where $\mathbb{I}$ is the identity operator on $\mathbb{H}$ and $\mathscr{C}$ is a positively-oriented simple rectifiable closed contour containing the spectrum $\sigma(\mathbb{P})$ in the interior domain. The operator $f(\mathbb{P})$ can also be defined by the following series:

$$f(\mathbb{P}) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \mathbb{P}^n$$

which converges in the normed topology (cf. [26]).

Motivated by earlier works on meromorphic functions by function theorists (see [2, 9, 10, 12, 15, 27–30], and certain studies on Hilbert space operators [31–33] in this paper we made an attempt to define the new subclass $\mathfrak{M}_{\varsigma,\varrho}^{\tau,\kappa,m}(\vartheta,\wp)$ of Σ_P, as given in Definition 1, related with the generalized operator $\mathscr{I}_{\varsigma,\varrho}^{\tau,\kappa,m}$.

Definition 1. For $0 \leqslant \vartheta < 1$ and $0 \leqslant \wp < 1$, a function $f \in \Sigma_p$ be given by (1.2) is said to be in the class $\mathfrak{M}_{\varsigma,\varrho}^{\tau,\kappa,m}(\vartheta,\wp)$, if

$$\left\| \frac{\mathscr{J}_{\wp}(\mathbb{P}) - 1}{\mathscr{J}_{\wp}(\mathbb{P}) + (1-2\vartheta)} \right\| < 1, \tag{1.17}$$

where

$$\mathscr{J}_{\wp}(\mathbb{P}) = \frac{\mathbb{P}(\mathscr{I}_{\varsigma,\varrho}^{\tau,\kappa,m} f(\mathbb{P}))'}{(\wp-1)\mathscr{I}_{\varsigma,\varrho}^{\tau,\kappa,m} f(\mathbb{P}) + \wp\mathbb{P}(\mathscr{I}_{\varsigma,\varrho}^{\tau,\kappa,m} f(\mathbb{P}))'}. \tag{1.18}$$

By fixing $\wp = 0$, we also define a new class of functions in Definition 2 and denote it by $\mathfrak{S}_{\varsigma,\varrho}^{\tau,\kappa}(\vartheta,\wp)$.

Definition 2. For $0 \leqslant \vartheta < 1$ and $0 \leqslant \wp < 1$, a function $f \in \Sigma_p$ be given by (1.2) is said to be in the class $\mathfrak{S}_{\varsigma,\varrho}^{\tau,\kappa}(\vartheta,\wp)$, if

$$\left\| \frac{\frac{\mathbb{P}(\mathscr{I}_{\varsigma,\varrho}^{\tau,\kappa,m} f(\mathbb{P}))'}{\mathscr{I}_{\varsigma,\varrho}^{\tau,\kappa,m} f(\mathbb{P})} - 1}{\frac{\mathbb{P}(\mathscr{I}_{\varsigma,\varrho}^{\tau,\kappa,m} f(\mathbb{P}))'}{\mathscr{I}_{\varsigma,\varrho}^{\tau,\kappa,m} f(\mathbb{P})} + (1-2\vartheta)} \right\| < 1. \tag{1.19}$$

The present paper aims to provide a systematic investigation of the various interesting properties like coefficient inequalities, growth and distortion inequalities, as well as closure results for f in the class $\mathfrak{M}_{\varsigma,\varrho}^{\tau,\kappa,m}(\vartheta,\wp)$ extensively. Properties of a certain integral operator and its inverse defined on the new class $\mathfrak{M}_{\varsigma,\varrho}^{\tau,\kappa,m}(\vartheta,\wp)$ are also discussed.

2. Coefficients Inequalities

Our first theorem gives a necessary and sufficient condition for a function $f \in \mathfrak{M}_{\varsigma,\varrho}^{\tau,\kappa,m}(\vartheta,\wp)$.

Theorem 1. *Let $f \in \Sigma_P$ be given by (1.2). Then $f \in \mathfrak{M}_{\varsigma,\varrho}^{\tau,\kappa,m}(\vartheta,\wp)$ if and only if*

$$\sum_{n=1}^{\infty} \{n - \vartheta(n\wp + \wp - 1)\}\, \mho_n a_n \leqslant 1 - \vartheta. \tag{2.1}$$

$\lhd$ Suppose f satisfies (2.1). Then for $\|z\| = \mathbb{P} = r\mathbb{I}$ we have

$$\begin{aligned} &\left\|\mathbb{P}(\mathscr{I}_{\varsigma,\varrho}^{\tau,\kappa,m} f(\mathbb{P}))' - \{(\wp-1)\mathscr{I}_{\varsigma,\varrho}^{\tau,\kappa,m} f(\mathbb{P}) + \wp\mathbb{P}(\mathscr{I}_{\varsigma,\varrho}^{\tau,\kappa,m} f(\mathbb{P}))'\}\right\| \\ &- \left\|\mathbb{P}f'(\mathbb{P}) + (1-2\vartheta)\left\{(\wp-1)\mathscr{I}_{\varsigma,\varrho}^{\tau,\kappa,m} f(\mathbb{P}) + \wp\mathbb{P}(\mathscr{I}_{\varsigma,\varrho}^{\tau,\kappa,m} f(\mathbb{P}))'\right\}\right\| \\ &= \left\|\sum_{n=1}^{\infty}(1-\wp)(n+1)\mho_n a_n \mathbb{P}^{n+1}\right\| \end{aligned}$$

$$-\left\|-2(1-\vartheta)+\sum_{n=1}^{\infty}\Big[\{1+(1-2\vartheta)\wp\}n+(1-2\vartheta)(\wp-1)\Big]\mho_n a_n\mathbb{P}^{n+1}\right\|$$

$$\leqslant\sum_{n=1}^{\infty}(1-\wp)(n+1)\mho_n a_n\|\mathbb{P}^{n+1}\|$$

$$-2(1-\vartheta)+\sum_{n=1}^{\infty}\Big[\{1+(1-2\vartheta)\wp\}n+(1-2\vartheta)(\wp-1)\Big]\mho_n a_n\|\mathbb{P}^{n+1}\|$$

$$=2\sum_{n=1}^{\infty}\{n-\vartheta(n\wp+\wp-1)\}\,\mho_n a_n r^{n+1}-2(1-\vartheta)\leqslant 0,\quad\text{by (2.1)}.$$

Hence, f satisfies (1.17), and $f\in\mathfrak{M}_{\varsigma,\varrho}^{\tau,\kappa,m}(\vartheta,\wp)$. Now to prove the converse, let $f\in\mathfrak{M}_{\varsigma,\varrho}^{\tau,\kappa,m}(\vartheta,\wp)$. We need only to show that each function f of the class $\mathfrak{M}_{\varsigma,\varrho}^{\tau,\kappa,m}(\vartheta,\wp)$ satisfies the coefficient inequality (2.1). Since $f\in\mathfrak{M}_{\varsigma,\varrho}^{\tau,\kappa,m}(\vartheta,\wp)$, we have by definition

$$\left\|\frac{\frac{\mathbb{P}(\mathscr{I}_{\varsigma,\varrho}^{\tau,\kappa,m}f(\mathbb{P}))'}{(\wp-1)\mathscr{I}_{\varsigma,\varrho}^{\tau,\kappa,m}f(\mathbb{P})+\wp\mathbb{P}(\mathscr{I}_{\varsigma,\varrho}^{\tau,\kappa,m}f(z))'}-1}{\frac{\mathbb{P}f'(\mathbb{P})}{(\wp-1)\mathscr{I}_{\varsigma,\varrho}^{\tau,\kappa,m}f(\mathbb{P})+\wp\mathbb{P}(\mathscr{I}_{\varsigma,\varrho}^{\tau,\kappa,m}f(\mathbb{P}))'}+(1-2\vartheta)}\right\|<1,\quad z\in\Delta.$$

That is

$$\left\|\frac{\sum_{n=1}^{\infty}(1-\wp)(n+1)\mho_n a_n\mathbb{P}^{n+1}}{-2(1-\vartheta)+\sum_{n=1}^{\infty}\left[\{1+(1-2\vartheta)\wp\}n+(1-2\vartheta)(\wp-1)\right]\mho_n a_n\mathbb{P}^{n+1}}\right\|\leqslant 1.$$

Since $|\operatorname{Re}(z)|\leqslant|z|=r$ for $z\in\mathbb{C}$ thus by taking $\mathbb{P}=r\mathbb{I}$ $(0<r<1)$, from the above inequality we have

$$\left[\frac{\sum_{n=1}^{\infty}(1-\wp)(n+1)\mho_n a_n r^{n+1}}{2(1-\vartheta)-\sum_{n=1}^{\infty}\left[\{1+(1-2\vartheta)\wp\}n+(1-2\vartheta)(\wp-1)\right]\mho_n a_n r^{n+1}}\right]\leqslant 1,$$

and letting $r\to 1^-$, yields the assertion (2.1) of Theorem 1. ▷

Fixing $\wp=0$, we get the following.

Corollary 2. *Let $f\in\Sigma_P$ be given by (1.2). Then $f\in\mathfrak{S}_{\varsigma,\varrho}^{\tau,\kappa}(\vartheta,\wp)$ if and only if*

$$\sum_{n=1}^{\infty}(n+\vartheta)\mho_n a_n\leqslant 1-\vartheta.$$

Our next result gives the coefficient estimates for functions in $\mathfrak{M}_{\varsigma,\varrho}^{\tau,\kappa,m}(\vartheta,\wp)$.

Theorem 2. *If $f\in\mathfrak{M}_{\varsigma,\varrho}^{\tau,\kappa,m}(\vartheta,\wp)$, then*

$$a_n\leqslant\frac{1-\vartheta}{\{n-\vartheta(n\wp+\wp-1)\}\,\mho_n},\qquad n=1,2,3,\dots$$

The result is sharp for the functions $F_n(z)$ given by

$$F_n(z)=\frac{1}{z}+\frac{1-\vartheta}{\{n-\vartheta(n\wp+\wp-1)\}\,\mho_n}z^n,\qquad n=1,2,3,\dots$$

◁ If $f\in\mathfrak{M}_{\varsigma,\varrho}^{\tau,\kappa,m}(\vartheta,\wp)$, then for each n we have

$$\{n-\vartheta(n\wp+\wp-1)\}\,\mho_n a_n\leqslant\sum_{n=1}^{\infty}\{n-\vartheta(n\wp+\wp-1)\}\,a_n\leqslant 1-\vartheta.$$

Therefore, we have

$$a_n\leqslant\frac{1-\vartheta}{\{n-\vartheta(n\wp+\wp-1)\}\,\mho_n}.$$

Since

$$F_n(z) = \frac{1}{z} + \frac{1-\vartheta}{\{n - \vartheta(n\wp + \wp - 1)\}\,\mho_n} z^n$$

satisfies the conditions of Theorem 1, $F_n(z) \in \mathfrak{M}_{\varsigma,\varrho}^{\tau,\kappa,m}(\vartheta,\wp)$ and the equality is attained for this function. ▷

For $\wp = 0$, we have the following corollary.

Corollary 3. *If $f \in \mathfrak{S}_{\varsigma,\varrho}^{\tau,\kappa}(\vartheta,\wp)$, then*

$$a_n \leqslant \frac{1-\vartheta}{(n+\vartheta)\mho_n}, \quad n = 1, 2, 3, \ldots.$$

Theorem 3. *If $f \in \mathfrak{M}_{\varsigma,\varrho}^{\tau,\kappa,m}(\vartheta,\wp)$, then*

$$\frac{1}{r} - \frac{1-\vartheta}{(1+\vartheta-2\vartheta\wp)\mho_1} r \leqslant \|f(\mathbb{P})\| \leqslant \frac{1}{r} + \frac{1-\vartheta}{(1+\vartheta-2\vartheta\wp)\mho_1} r, \quad \mathbb{P} = r \ (0 < r < 1).$$

The result is sharp for

$$f(z) = \frac{1}{z} + \frac{1-\vartheta}{(1+\vartheta-2\vartheta\wp)\mho_1} z, \tag{2.2}$$

where $\mho_1$ as given in (1.15).

◁ Since $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n$, we have

$$\|f(\mathbb{P})\| \leqslant \frac{1}{r} + \sum_{n=1}^{\infty} a_n r^n \leqslant \frac{1}{r} + r \sum_{n=1}^{\infty} a_n.$$

Taking into account the inequality

$$\sum_{n=1}^{\infty} a_n \leqslant \frac{1-\vartheta}{(1+\vartheta-2\vartheta\wp)\mho_1}$$

we arrive at an estimate

$$\|f(\mathbb{P})\| \leqslant \frac{1}{r} + \frac{1-\vartheta}{(1+\vartheta-2\vartheta\wp)\mho_1} r.$$

Similarly

$$\|f(\mathbb{P})\| \geqslant \frac{1}{r} - \frac{1-\vartheta}{(1+\vartheta-2\vartheta\wp)\mho_1} r.$$

The result is sharp for $f(z) = \frac{1}{z} + \frac{1-\vartheta}{(1+\vartheta-2\vartheta\wp)\mho_1} z$. ▷

Similarly we have the following:

Theorem 4. *If $f \in \mathfrak{M}_{\varsigma,\varrho}^{\tau,\kappa,m}(\vartheta,\wp)$, then*

$$\frac{1}{r^2} - \frac{1-\vartheta}{(1+\vartheta-2\vartheta\wp)\mho_1} \leqslant \|f'(\mathbb{P})\| \leqslant \frac{1}{r^2} + \frac{1-\vartheta}{(1+\vartheta-2\vartheta\wp)\mho_1}, \quad \mathbb{P} = r \ (0 < r < 1).$$

The result is sharp for the function given by (2.2).

3. Radius of Starlikeness

The radii of starlikeness and convexity are given by the following theorems for the class $\mathfrak{M}_{\varsigma,\varrho}^{\tau,\kappa,m}(\vartheta,\wp)$.

Theorem 5. *Let the function f belong to the class $\mathfrak{M}_{\varsigma,\varrho}^{\tau,\kappa,m}(\vartheta,\wp)$. Then f is meromorphically starlike of order ρ $(0 \leqslant \rho < 1)$ in $|z| < r_1(\vartheta,\wp,\rho)$, where*

$$r_1(\vartheta,\wp,\rho) = \inf_n \left[\frac{(1-\rho)[n - \vartheta(n\wp + \wp - 1)]\mho_n}{(n+\rho)(1-\vartheta)}\right]^{\frac{1}{n+1}}, \quad n \geqq 1. \tag{3.1}$$

$\lhd$ Let the function $f \in M_m^l(\wp,\vartheta)$ be of the form (1.2). If $0 < r \leqslant r_1(\vartheta,\wp,\rho)$, then by (3.1)

$$r^{n+1} \leqslant \frac{(1-\rho)\big[n-\vartheta(n\wp+\wp-1)\big]\mho_n}{(n+\rho)(1-\vartheta)} \tag{3.2}$$

for all $n \in \mathbb{N}$. From (3.2) we get $\frac{n+\rho}{1-\rho}r^{n+1} \leqslant \frac{\big[n-\vartheta(n\wp+\wp-1)\big]\mho_n}{1-\vartheta}$ for all $n \in \mathbb{N}$, thus

$$\sum_{n=1}^{\infty}\frac{n+\rho}{1-\rho}a_n r^{n+1} \leqslant \sum_{n=1}^{\infty}\frac{\big[n-\vartheta(n\wp+\wp-1)\big]\mho_n}{1-\vartheta}a_n \leqslant 1 \tag{3.3}$$

because of (2.1). If $f \in \Sigma_P$, then by Lemma 1 the function f is meromorphically starlike of order ρ in $|z| < r$ if and only if

$$\sum_{n=1}^{\infty}(n+\rho)a_n r^{n+1} \leqslant 1-\rho. \tag{3.4}$$

Therefore, (3.3) and (3.4) give that f is meromorphically starlike of order ρ in $|z| < r \leqslant r_1(\vartheta,\wp,\rho)$. $\rhd$

Suppose that there exists a number $\widetilde{r}$, $\widetilde{r} > r_1(\vartheta,\wp,\rho)$, such that each $f \in \mathfrak{M}_{\varsigma,\varrho}^{\tau;\kappa}(\wp,\vartheta)$ is meromorphically starlike of order ρ in $|z| < \widetilde{r} \leqslant 1$. The function

$$f(z) = \frac{1}{z} + \frac{1-\vartheta}{[n-\vartheta(n\wp+\wp-1)]\mho_n}z^n$$

is in the class $\mathfrak{M}_{\varsigma,\varrho}^{\tau;\kappa}(\wp,\vartheta)$, thus it should satisfy (3.4) with $\widetilde{r}$:

$$\sum_{n=1}^{\infty}(n+\rho)a_n \widetilde{r}^{n+1} \leqslant 1-\rho, \tag{3.5}$$

while the left–hand side of (3.5) becomes

$$(n+\rho)\frac{(1-\vartheta)}{[n-\vartheta(n\wp+\wp-1)]\mho_1}\widetilde{r}^{n+1} > (n+\rho)\frac{(1-\vartheta)}{[n-\vartheta(n\wp+\wp-1)]\mho_1}$$
$$\times\frac{(1-\rho)[n-\vartheta(n\wp+\wp-1)]\mho_n}{(n+\rho)(1-\vartheta)} = 1-\rho$$

which contradicts (3.5). Therefore, the number $r_1(\vartheta,\wp,\rho)$ in Theorem 5 cannot be replaced with a grater number. This means that $r_1(\vartheta,\wp,\rho)$ is so called radius of meromorphically starlikness of order ρ for the class $\mathfrak{M}_{\varsigma,\varrho}^{\tau;\kappa}(\wp,\vartheta)$.

Remark 1. The above results give an improvement or better bound for order of starlikeness for $f \in \mathfrak{M}_{\varsigma,\varrho}^{\tau;\kappa,\gamma}(\vartheta,\wp)$ compared to the results given in [32, 33].

4. Neighborhoods for the Class $\mathfrak{M}_{\varsigma,\varrho}^{\tau;\kappa,\gamma}(\vartheta,\wp)$

In this section, we determine the neighborhood for the class $\mathfrak{M}_{\varsigma,\varrho}^{\tau;\kappa,\gamma}(\vartheta,\wp)$, which we define as follows:

Definition 3. A function $f \in \Sigma_p$ is said to be in the class $\mathfrak{M}_{\varsigma,\varrho}^{\tau;\kappa,\gamma}(\vartheta,\wp)$ if there exists a function $g \in \mathfrak{M}_{\varsigma,\varrho}^{\tau;\kappa,m}(\vartheta,\wp)$ such that

$$\left\|\frac{f(\mathbb{P})}{g(\mathbb{P})}-1\right\| < 1-\gamma, \qquad (z \in \Delta,\ 0 \leqslant \gamma < 1). \tag{4.1}$$

Following the earlier works on neighborhoods of analytic functions by Goodman [34] and Ruscheweyh [35], we define the δ-neighborhood of a function $f \in \Sigma_p$ by

$$N_\delta(f) := \left\{ g \in \Sigma_p : \ g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n \text{ and } \sum_{n=1}^{\infty} n|a_n - b_n| \leqslant \delta \right\}. \tag{4.2}$$

Theorem 6. *If* $g \in \mathfrak{M}_{\varsigma,\varrho}^{\tau,\kappa,m}(\vartheta, \wp)$ *and*

$$\gamma = 1 - \frac{\delta(1 + \vartheta - 2\vartheta\wp)\mho_1}{2\vartheta(1-\wp)}, \tag{4.3}$$

then $N_\delta(g) \subset \mathfrak{M}_{\varsigma,\varrho}^{\tau,\kappa,\gamma}(\vartheta, \wp)$.

◁ Let $f \in N_\delta(g)$. Then we find from (4.2) that

$$\sum_{n=1}^{\infty} n|a_n - b_n| \leqslant \delta, \tag{4.4}$$

which implies the coefficient inequality

$$\sum_{n=1}^{\infty} |a_n - b_n| \leqslant \delta, \quad n \in \mathbb{N}. \tag{4.5}$$

Since $g \in \mathfrak{M}_{\varsigma,\varrho}^{\tau,\kappa,m}(\vartheta, \wp)$, we have (cf. equation (2.1))

$$\sum_{n=1}^{\infty} b_n \leqslant \frac{1-\vartheta}{(1 + \vartheta - 2\vartheta\wp)\mho_1}, \tag{4.6}$$

so that

$$\left\| \frac{f(\mathbb{P}))}{g(\mathbb{P})} - 1 \right\| < \frac{\sum_{n=1}^{\infty} |a_n - b_n|}{1 - \sum_{n=1}^{\infty} b_n} = \frac{\delta(1 + \vartheta - 2\vartheta\wp)\mho_1}{2\vartheta(1-\wp)} = 1 - \gamma,$$

provided γ is given by (4.3). Hence, by definition, $f \in \mathfrak{M}_{\varsigma,\varrho}^{\tau,\kappa,\gamma}(\vartheta, \wp)$ for γ given by (4.3), which completes the proof. ▷

5. Closure Theorems

Let the functions $F_k(z)$ be given by

$$F_k(z) = \frac{1}{z} + \sum_{n=1}^{\infty} f_{n,k} z^n, \quad k = 1, 2, \ldots, m. \tag{5.1}$$

We shall prove the following closure theorems for the class $\mathfrak{M}_{\varsigma,\varrho}^{\tau,\kappa,m}(\vartheta, \wp)$.

Theorem 7. *Let the function* $F_k(z)$ *defined by* (5.1) *be in the class* $\mathfrak{M}_{\varsigma,\varrho}^{\tau,\kappa,m}(\vartheta, \wp)$ *for every* $k = 1, 2, \ldots, m$. *Then the function* $f(z)$ *defined by*

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n, \quad a_n \geqslant 0$$

belongs to the class $\mathfrak{M}_{\varsigma,\varrho}^{\tau,\kappa,m}(\vartheta, \wp)$, *whenever* $a_n = \frac{1}{m}\sum_{k=1}^{m} f_{n,k}$, $n = 1, 2, \ldots$

◁ Since $F_n(z) \in \mathfrak{M}_{\varsigma,\varrho}^{\tau,\kappa,m}(\vartheta,\wp)$, it follows from Theorem 1 that

$$\sum_{n=1}^{\infty} \big\{n-\vartheta(n\wp+\wp-1)\big\}\mho_n f_{n,k} \leqslant 1-\vartheta \tag{5.2}$$

for every $k=1,2,\dots,m$. Hence

$$\sum_{n=1}^{\infty} \{n-\vartheta(n\wp+\wp-1)\}\,\mho_n a_n = \sum_{n=1}^{\infty} \{n-\vartheta(n\wp+\wp-1)\}\,\mho_n \left(\frac{1}{m}\sum_{k=1}^{m} f_{n,k}\right)$$
$$= \frac{1}{m}\sum_{k=1}^{m}\left(\sum_{n=1}^{\infty} \{n-\vartheta(n\wp+\wp-1)\}\,\mho_n f_{n,k}\right) \leqslant 1-\vartheta.$$

By Theorem 1 we arrive at the required conclusion $f(z) \in \mathfrak{M}_{\varsigma,\varrho}^{\tau,\kappa,m}(\vartheta,\wp)$. ▷

Theorem 8. *The class $\mathfrak{M}_{\varsigma,\varrho}^{\tau,\kappa,m}(\vartheta,\wp)$ is closed under convex linear combination.*

◁ Let the function $F_k(z)$ given by (5.1) be in the class $\mathfrak{M}_{\varsigma,\varrho}^{\tau,\kappa,m}(\vartheta,\wp)$. Then it is enough to show that the function

$$H(z) = \nu F_1(z) + (1-\nu)F_2(z), \quad 0 \leqslant \wp \leqslant 1,$$

is also in the class $\mathfrak{M}_{\varsigma,\varrho}^{\tau,\kappa,m}(\vartheta,\wp)$. Since for $0 \leqslant \nu \leqslant 1$,

$$H(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \big[\nu f_{n,1} + (1-\nu) f_{n,2}\big] z^n,$$

we observe that

$$\sum_{n=1}^{\infty} \{n-\vartheta(n\wp+\wp-1)\}\,\mho_n[\nu f_{n,1} + (1-\nu) f_{n,2}]$$
$$= \nu\sum_{n=1}^{\infty} \{n-\vartheta(n\wp+\wp-1)\}\,\mho_n f_{n,1} + (1-\nu)\sum_{n=1}^{\infty} \{n-\vartheta(n\wp+\wp-1)\}\,\mho_n f_{n,2} \leqslant 1-\vartheta.$$

By Theorem 1, we have $H(z) \in \mathfrak{M}_{\varsigma,\varrho}^{\tau,\kappa,m}(\vartheta,\wp)$. ▷

Theorem 9. *Let $F_0(z) = \frac{1}{z}$ and $F_n(z) = \frac{1}{z} + \frac{1-\vartheta}{\{n-\vartheta(n\wp+\wp-1)\}\mho_n} z^n$ for $n=1,2,\dots$ Then $f(z) \in \mathfrak{M}_{\varsigma,\varrho}^{\tau,\kappa,m}(\vartheta,\wp)$ if and only if $f(z)$ can be expressed in the form $f(z) = \sum_{n=0}^{\infty} \nu_n F_n(z)$, where $\nu_n \geqslant 0$ and $\sum_{n=0}^{\infty} \nu_n = 1$.*

◁ Let

$$f(z) = \sum_{n=0}^{\infty} \nu_n F_n(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{\nu_n(1-\vartheta)}{\{n-\vartheta(n\wp+\wp-1)\}\,\mho_n} z^n.$$

Then

$$\sum_{n=1}^{\infty} \nu_n \frac{1-\vartheta}{\{n-\vartheta(n\wp+\wp-1)\}\,\mho_n}\,\frac{\{n-\vartheta(n\wp+\wp-1)\}\,\mho_n}{(1-\vartheta)} = \sum_{n=1}^{\infty} \nu_n = 1-\nu_0 \leqslant 1.$$

By Theorem 1, we have $f(z) \in \mathfrak{M}_{\varsigma,\varrho}^{\tau,\kappa,m}(\vartheta,\wp)$.

Conversely, let $f(z) \in \mathfrak{M}_{\varsigma,\varrho}^{\tau,\kappa,m}(\vartheta,\wp)$. In view of Theorem 2, we have

$$a_n \leqslant \frac{1-\vartheta}{\{n-\vartheta(n\wp+\wp-1)\}\,\mho_n}, \quad n=1,2,\dots,$$

and we may take

$$\nu_n = \frac{\{n - \vartheta(n\wp + \wp - 1)\}\, \mho_n}{1-\vartheta} a_n, \quad n = 1, 2, \dots,$$

and $\nu_0 = 1 - \sum_{n=1}^{\infty} \nu_n$. Then $f(z) = \sum_{n=0}^{\infty} \nu_n F_n(z)$. ▷

6. Partial Sums

Silverman [36] determined sharp lower bounds on the real part of the quotients between the normalized starlike or convex functions and their sequences of partial sums. As a natural extension, it would be interesting to look for results similar to those by Silverman for meromorphic univalent functions. In this section, motivated essentially by the work of Silverman [36] and Cho and Owa [10] we will investigate the ratio of a function of the form

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n, \tag{6.1}$$

to its sequence of partial sums

$$f_1(z) = \frac{1}{z} \text{ and } f_k(z) = \frac{1}{z} + \sum_{n=1}^{k} a_n z^n, \tag{6.2}$$

when the coefficients are sufficiently small to satisfy the condition analogous to

$$\sum_{n=1}^{\infty} \{n - \vartheta(n\wp + \wp - 1)\}\, \mho_n \; a_n \leqslant 1 - \vartheta.$$

For the sake of brevity we rewrite it as

$$\sum_{n=1}^{\infty} \Lambda_n |a_n| \leqslant 1 - \vartheta, \tag{6.3}$$

where

$$\Lambda_n := [n - \vartheta(n\wp + \wp - 1)]\mho_n. \tag{6.4}$$

More precisely we will determine sharp lower bounds for $\mathrm{Re}\{f(z)/f_k(z)\}$ and $\mathrm{Re}\{f_k(z)/f(z)\}$. In this connection we make use of the well known results that $\mathrm{Re}\left\{\frac{1+w(z)}{1-w(z)}\right\} > 0$ $(z \in \Delta)$ if and only if

$$\omega(z) = \sum_{n=1}^{\infty} c_n z^n$$

satisfies the inequality $|\omega(z)| \leqslant |z|$. Unless otherwise stated, we will assume that f is of the form (1.2) and its sequence of partial sums is denoted by

$$f_k(z) = \frac{1}{z} + \sum_{n=1}^{k} a_n z^n.$$

Theorem 10. *Let* $f(z) \in \Sigma_P(\vartheta, \wp)$ *given by* (6.1) *satisfiy condition* (2.1). *Then*

$$Re\left\{\frac{f(z)}{f_k(z)}\right\} \geqslant \frac{\Lambda_{k+1}(\wp,\vartheta) - 1 + \vartheta}{\Lambda_{k+1}(\wp,\vartheta)}, \quad z \in U, \tag{6.5}$$

where

$$\Lambda_n(\wp,\vartheta) \geqslant \begin{cases} 1-\vartheta, & n=1,2,3,\ldots,k\,; \\ \Lambda_{k+1}(\wp,\vartheta), & n=k+1,k+2,\ldots \end{cases} \tag{6.6}$$

The result (6.5) *is sharp with the function given by*

$$f(z)=\frac{1}{z}+\frac{1-\vartheta}{\Lambda_{k+1}(\wp,\vartheta)}z^{k+1}. \tag{6.7}$$

◁ Define the function $w(z)$ by

$$\frac{1+w(z)}{1-w(z)}=\frac{\Lambda_{k+1}(\wp,\vartheta)}{1-\vartheta}\left[\frac{f(z)}{f_k(z)}-\frac{\Lambda_{k+1}(\wp,\vartheta)-1+\vartheta}{\Lambda_{k+1}(\wp,\vartheta)}\right]$$

$$=\frac{1+\sum\limits_{n=1}^{k}a_nz^{n+1}+\left(\frac{\Lambda_{k+1}(\wp,\vartheta)}{1-\vartheta}\right)\sum\limits_{n=k+1}^{\infty}a_nz^{n+1}}{1+\sum\limits_{n=1}^{k}a_nz^{n+1}}. \tag{6.8}$$

It suffices to show that $|w(z)|\leqslant 1$. Now, from (6.8) we can write

$$w(z)=\frac{\left(\frac{\Lambda_{k+1}(\wp,\vartheta)}{1-\vartheta}\right)\sum\limits_{n=k+1}^{\infty}a_nz^{n+1}}{2+2\sum\limits_{n=1}^{k}a_nz^{n+1}+\left(\frac{\Lambda_{k+1}(\wp,\vartheta)}{1-\vartheta}\right)\sum\limits_{k=n+1}^{\infty}a_nz^{n+1}}.$$

Next we estimate

$$|w(z)|\leqslant\frac{\left(\frac{\Lambda_{k+1}(\wp,\vartheta)}{1-\vartheta}\right)\sum\limits_{k=n+1}^{\infty}|a_n|}{2-2\sum\limits_{n=1}^{k}|a_n|-\left(\frac{\Lambda_{k+1}(\wp,\vartheta)}{1-\vartheta}\right)\sum\limits_{n=k+1}^{\infty}|a_n|}.$$

Now $\|w(\mathbb{P})\|\leqslant 1$, if

$$2\left(\frac{\Lambda_{k+1}(\wp,\vartheta)}{1-\vartheta}\right)\sum_{n=k+1}^{\infty}|a_n|\leqslant 2-2\sum_{n=1}^{k}|a_n|$$

or, equivalently,

$$\sum_{n=1}^{k}|a_n|+\frac{\Lambda_{k+1}(\wp,\vartheta)}{1-\vartheta}\sum_{n=k+1}^{\infty}|a_n|\leqslant 1.$$

Due to the condition (2.1), it is sufficient to show that

$$\sum_{n=1}^{k}|a_n|+\frac{\Lambda_{k+1}(\wp,\vartheta)}{1-\vartheta}\sum_{n=k+1}^{\infty}|a_n|\leqslant\sum_{n=1}^{\infty}\frac{\Lambda_n(\wp,\vartheta)}{1-\vartheta}|a_n|$$

which is equivalent to

$$\sum_{n=1}^{k}\left(\frac{\Lambda_n(\wp,\vartheta)-1+\vartheta}{1-\vartheta}\right)|a_n|+\sum_{n=k+1}^{\infty}\left(\frac{\Lambda_n(\wp,\vartheta)-\Lambda_{k+1}(\wp,\vartheta)}{1-\vartheta}\right)|a_n|\geqslant 0.$$

To see that the function given by (6.7) gives the sharp result, we observe that for $z = re^{i\pi/k}$

$$\frac{f(z)}{f_k(z)} = 1 + \frac{1-\vartheta}{\Lambda_{k+1}(\wp,\vartheta)} z^n \to 1 - \frac{1-\vartheta}{\Lambda_{k+1}(\wp,\vartheta)} = \frac{\Lambda_{k+1}(\wp,\vartheta) - 1 + \vartheta}{\Lambda_{k+1}(\wp,\vartheta)}, \quad \text{when } r \to 1^-,$$

which shows the bound (6.5) is the best possible for each $k \in \mathbb{N}$. ▷

7. Integral Operators

In this section, we consider integral transforms of functions in the class $\mathfrak{M}_{\varsigma,\varrho}^{\tau,\kappa,m}(\vartheta,\wp)$.

Theorem 11. *Let the function $f(z)$ given by (1) be in $\mathfrak{M}_{\varsigma,\varrho}^{\tau,\kappa,m}(\vartheta,\wp)$. Then the integral operator*

$$F(z) = c\int_0^1 u^c f(uz)\,du, \quad 0 < u \leqslant 1,\ 0 < c < \infty,$$

is in $\mathfrak{M}_{\varsigma,\varrho}^{\tau,\kappa}(\delta,\wp)$, where

$$\delta = \frac{(c+2)\{1+\vartheta-2\vartheta\wp\} - c(1-\vartheta)}{c(1-\vartheta)\{1-2\wp\} + (1+\vartheta)\{1-2\wp\}(c+2)}.$$

The result is sharp for the function $f(z) = \frac{1}{z} + \frac{1-\vartheta}{\{1+\vartheta-2\vartheta\wp\}\mho_1} z$.

◁ Let $f(z) \in \mathfrak{M}_{\varsigma,\varrho}^{\tau,\kappa,m}(\vartheta,\wp)$. Then

$$F(z) = c\int_0^1 u^c f(uz)\,du = c\int_0^1 \left(\frac{u^{c-1}}{z} + \sum_{n=1}^{\infty} f_n u^{n+c} z^n\right) du = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{c}{c+n+1} f_n z^n.$$

It is sufficient to show that

$$\sum_{n=1}^{\infty} \frac{c\{n-\delta(n\wp+\wp-1)\}\mho_n}{(c+n+1)(1-\delta)} a_n \leqslant 1. \tag{7.1}$$

Since $f \in \mathfrak{M}_{\varsigma,\varrho}^{\tau,\kappa,m}(\vartheta,\wp)$, we have

$$\sum_{n=1}^{\infty} \frac{\{n-\vartheta(n\wp+\wp-1)\}}{1-\vartheta} \mho_n a_n \leqslant 1.$$

Note that (7.1) is satisfied, if

$$\frac{c\{n-\delta(n\wp+\wp-1)\}\mho_n}{(c+n+1)(1-\delta)} \leqslant \frac{\{n-\vartheta(n\wp+\wp-1)\}\mho_n}{(1-\vartheta)}.$$

Rewriting the inequality, we have

$$c\{n-\delta(n\wp+\wp-1)\}(1-\vartheta) \leqslant (c+n+1)(1-\delta)\{n+\vartheta-\vartheta\wp(1+n)\}\mho_n.$$

Solving for δ, we have

$$\delta \leqslant \frac{(c+n+1)\{n-\vartheta(n\wp+\wp-1)\} - cn(1-\vartheta)}{c(1-\vartheta)\{1-\wp(1+n)\} + \{(n-\vartheta(n\wp+\wp-1)\}(c+n+1)} = F(n).$$

A simple computation will show that $F(n)$ is increasing and $F(n) \geqslant F(1)$. Using this, the results follows. ▷

For the choice of $\wp = 0$, we have the following result of Uralegaddi and Ganigi [9].

Corollary 4. *Let the function $f(z)$ defined by (1) be in $\Sigma_p^*(\vartheta)$. Then the integral operator*

$$F(z) = c\int_0^1 u^c f(uz)\,du, \quad 0 < u \leqslant 1,\ 0 < c < \infty,$$

is in $\Sigma_p^*(\delta)$, where $\delta = \frac{1+\vartheta+c\vartheta}{1+\vartheta+c}$. The result is sharp for the function

$$f(z) = \frac{1}{z} + \frac{1-\vartheta}{1+\vartheta}z.$$

Also we have the following:

Theorem 12. *Let $f(z)$, given by (1), be in $\mathfrak{M}_{\varsigma,\varrho}^{\tau,\kappa,m}(\vartheta,\wp)$,*

$$F(z) = \frac{1}{c}\big[(c+1)f(z) + zf'(z)\big] = \frac{1}{z} + \sum_{n=1}^{\infty}\frac{c+n+1}{c}f_n z^n, \qquad c > 0. \tag{7.2}$$

Then $F(z)$ is in $\mathfrak{M}_{\varsigma,\varrho}^{\tau,\kappa,m}(\vartheta,\wp)$ for $|z| \leqslant r(\vartheta,\wp,\beta)$, where

$$r(\vartheta,\wp,\beta) = \inf_n \left(\frac{c(1-\beta)\left\{n-\vartheta(n\wp+\wp-1)\right\}}{(1-\vartheta)(c+n+1)\left\{n-\beta(n\wp+\wp-1)\right\}}\right)^{\frac{1}{n+1}}, \quad n = 1,2,3,\dots$$

The result is sharp for the function $f_n(z) = \frac{1}{z} + \frac{1-\vartheta}{\{n-\vartheta(n\wp+\wp-1)\}}z^n$, $\quad n = 1,2,3,\dots$

$\lhd$ Let $w = \frac{zf'(z)}{(\wp-1)f(z)+\wp zf'(z)}$. Then it is sufficient to show that

$$\left\|\frac{w-1}{w+1-2\beta}\right\| < 1.$$

A computation shows that this is satisfied, if

$$\sum_{n=1}^{\infty}\frac{\left\{n-\beta(n\wp+\wp-1)\right\}(c+n+1)}{(1-\beta)c}a_n\mho_n\|\mathbb{P}\|^{n+1} \leqslant 1. \tag{7.3}$$

Since $f \in \mathfrak{M}_{\varsigma,\varrho}^{\tau,\kappa,m}(\vartheta,\wp)$, by Theorem 1, we have

$$\sum_{n=1}^{\infty}\frac{(n-\vartheta(n\wp+\wp-1))\,\mho_n}{1-\vartheta}a_n \leqslant 1.$$

The equation (7.3) is satisfied, if

$$\frac{\left\{n-\beta(n\wp+\wp-1)\right\}(c+n+1)}{(1-\beta)c}\mho_n a_n|z|^{n+1} \leqslant \frac{\left\{n-\vartheta(n\wp+\wp-1)\right\}\mho_n a_n}{1-\vartheta}.$$

Solving for $|z|$, we get the result. $\rhd$

For the choice of $\wp = 0$, we have the following result of Uralegaddi and Ganigi [9].

Corollary 5. *Let the function $f(z)$ defined by (1) be in $\Sigma_p^*(\vartheta)$ and $F(z)$ given by (7.2). Then $F(z)$ is in $\Sigma_p^*(\vartheta)$ for $|z| \leqslant r(\vartheta,\beta)$, where*

$$r(\vartheta,\beta) = \inf_n\left(\frac{c(1-\beta)(n+\vartheta)}{(1-\vartheta)(c+n+1)(n+\beta)}\right)^{\frac{1}{n+1}}, \quad n = 1,2,3,\dots$$

The result is sharp for the function $f_n(z) = \frac{1}{z} + \frac{1-\vartheta}{n+\vartheta}z^n$, $n = 1,2,3,\dots$

Conclusion. The interplay of geometry and analysis signifies a vital aspect of the research in the complex functions theory. The rapid progress in this area is directly related to the relationship that exists between the analytic structure and the geometric behavior of functions. In the current study, we introduced a new class of meromorphic functions that is related to the Mittag-Leffler function based on the Hilbert space operator, and we found some sufficient and necessary conditions regarding the properties of this subclass. For further research we are intended to study certain classes related to functions with respect to fixed second coefficients associated with Mittag-Leffler functions and majorization results.

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О НОВОМ КЛАССЕ МЕРОМОРФНЫХ ФУНКЦИЙ, АССОЦИИРОВАННОМ С ФУНКЦИЕЙ МИТТАГ-ЛЕФФЛЕРА

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Аннотация. Функция Миттаг-Леффлера естественным образом возникает при решении дифференциальных и интегральных уравнений дробного порядка, и особенно, при изучении дробного обобщения кинетического уравнения, случайных блужданий, полетов Леви, супердиффузионного переноса и при изучении сложных систем. В настоящем исследовании авторы определяют новый класс $\mathfrak{M}_{\varsigma,\varrho}^{\tau,\kappa}(\vartheta,\wp)$ мероморфных функций, определенных в проколотом единичном круге $\Delta^* := \{z \in \mathbb{C} : 0 < |z| < 1\}$ на основе функции Миттаг-Леффлера. Подробно обсуждаются его характерные свойства, такие как коэффициентные неравенства, неравенства роста и искажения, а также результаты замыкания для $f \in \mathfrak{M}_{\varsigma,\varrho}^{\tau,\kappa}(\vartheta,\wp)$. Рассматриваются свойства некоторого интегрального оператора и его обратного, определенного на классе $\mathfrak{M}_{\varsigma,\varrho}^{\tau,\kappa}(\vartheta,\wp)$. Получены коэффициентные неравенства, неравенства роста и искажения, а также результаты замыкания. Установлены также некоторые результаты, касающиеся окрестностей и частичных сумм мероморфных функций в этом новом классе. Указаны некоторые новые подклассы и характеристические их свойства, специализируя параметры, которые являются новыми и не изучались ранее в связи с функциями Миттаг-Леффлера.

Ключевые слова: мероморфные функции, звездообразная функция, свертка, положительные коэффициенты, коэффициентные неравенства, интегральный оператор, функция Миттаг-Леффлера, оператор Гильбертова пространства.

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