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SUBGROUPS GENERATED BY A PAIR OF 2-TORI IN $\mathrm{GL}(4, K)$, II

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Abstract. The present paper is the next in a large series of works devoted to the geometry of microweight tori in the Chevalley groups. Namely, we describe the subgroups generated by a pair of 2-tori in $\mathrm{GL}(4, K)$. Recall that 2-tori in $\mathrm{GL}(n, K)$ are the subgroups conjugate to the diagonal subgroup of the following form $\mathrm{diag}(\varepsilon, \varepsilon, 1, \ldots, 1)$. In one of the previous work we proved the reduction theorem for the pairs of m-tori. It follows that any pair of 2-tori can be embedded in $\mathrm{GL}(6, K)$ by simultaneous conjugation. The orbit of a pair of 2-tori (X, Y) is called the orbit in $\mathrm{GL}(n, K)$, if the pair (X, Y) is embedded in $\mathrm{GL}(n, K)$ by simultaneous conjugation and it can not be embedded in $\mathrm{GL}(n-1, K)$. Here n can take values 3, 4, 5 and 6. The most difficult and general case is the case of $\mathrm{GL}(4, K)$. In the article we describe spans in $\mathrm{GL}(4, K)$, corresponding to degenerate orbits.

Keywords: general linear group, unipotent root subgroups, semisimple root subgroups, m-tori, diagonal subgroup.

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1. Introduction

The present paper is the next in a large series of works dedicated to the geometry of microweight tori and long root tori in the Chevalley groups that was announced in [1]. Namely we describe the subgroups generated by a pair of 2-tori in $\mathrm{GL}(4, K)$ corresponding to degenerate cases in the sense below.

Recall that 2-tori in $\mathrm{GL}(n, K)$ are the subgroups conjugate to the diagonal subgroup of the following form

$$\{\mathrm{diag}(\varepsilon, \varepsilon, 1, \ldots, 1),\ \varepsilon \in K^*\}.$$

From the general theory viewpoint 2-tori are microweight tori corresponding to the fundamental weight $\overline{\omega}_2$ in the extended Chevalley group of type A_{n-1}.

In [2] we proved the reduction theorem for the pairs of m-tori. It follows from it that any pair of 2-tori (X, Y) can be embedded in $\mathrm{GL}(6, K)$ by simultaneous conjugation. We call an orbit of a pair of 2-tori (X, Y) the orbit in $\mathrm{GL}(n, K)$, if the pair (X, Y) is embedded in $\mathrm{GL}(n, K)$ by simultaneous conjugation and it can not be embedded in $\mathrm{GL}(n-1, K)$. It follows from the reduction theorem that n can take values 3, 4, 5 or 6.

The orbits of 2-tori in $\mathrm{GL}(3,K)$ coincides with the orbits of 1-tori and are described in [3] (see also Lemma 1 [2]). The orbits and spans of 2-tori in $\mathrm{GL}(6,K)$ were classified in [2]. In paper [4] we described the orbits and spans of 2-tori in $\mathrm{GL}(5,K)$.

The case of $\mathrm{GL}(4,K)$ is the most difficult and requires cumbersome calculations. In [5] we classified the orbits of 2-tori in $\mathrm{GL}(4,K)$. It turned out that there are more than 80 types of such orbits. Of course in many cases pairs of 2-tori belongning to the different orbits have the same span. However calculations of all spans take a lot of pages. So we divided their into two papers. In this paper we treat degenerate cases, and in the next paper we consider undegenerate cases.

The context of this problem and many references reader can find in the surveys [1, 6, 7] and in the detailed introduction of [4].

The idea of this research belongs to wonderful mathematician and person Nikolai Aleksandrovich Vavilov. The starting point of it was his work [3]. The next three papers were written by N. A. Vavilov jointly with the first author. Unfortunately N. A. Vavilov passed away. The authors are very grateful to him for setting the problem and numerous inspiring discussions.

The authors are very pleased to publish this paper in the journal devoted to anniversary of professor V. A. Koibaev. The first author met him in the early 90s and remembered him as a very benevolence and responsiveness person.

2. Notation

This paper is a direct continuation of [5] and we use the same notation as before. But for reader's convenience we recall them here.

Let K be a field and $K^* = K\backslash\{0\}$ be the multiplicative group of it. Further, $G = \mathrm{GL}(n,K)$ is the general linear group of degree n over K. By $D = \mathrm{D}(n,K)$ we denote the subgroup of diagonal matrices in G, and $N = \mathrm{N}(n,K)$ denotes the subgroup of monomial matrices in G.

The quotient group N/D is isomorphic to S_n, the symmetric group on n letters. Denote by $W = \mathrm{W}_n$ the group of permutation matrices in G. We identify S_n and W_n via the isomorphism $\pi \mapsto w_\pi$, where w_π is the matrix whose entry in the position (i,j) is $\delta_{i,\pi j}$.

Let $V = K^n$ be the *right* vector space of columns of height n over K. Usually we identify a matrix $g \in G$ with the corresponding linear map of the space K^n. Here g acts *on the left*. To stress that we are using this geometric viewpoint, in such cases we call elements of G *transformations*.

By $e_1,\dots,e_n$ we denote the standard base of K^n. Here e_i is the column, whose i-th component equals 1, whereas all other components are equal to 0. The dual space $V^* = {}^nK$ is *left* vector space of rows of length n. By $f_1,\dots,f_n$ we denote the standard base of ${}^nK$. It is dual to $e_1,\dots,e_n$ with respect to the standard pairing, $V^*\times V \longrightarrow K$.

Denote by e_{ij} a standard matrix unit, i. e. the matrix whose entry in the position (i,j) is 1 and all the remaining entries are zeroes. Next, $x_{ij}(\xi) = e + \xi e_{ij}$ for $\xi \in K$ and $1 \leqslant i \neq j \leqslant n$ denotes *elementary transvection*. For given $i \neq j$ we consider the corresponding unipotent *root subgroup* $X_{ij} = \{x_{ij}(\xi), \xi \in K\}$. The subgroup $E(n,K)$ of G, generated by all X_{ij}, $1 \leqslant i \neq j \leqslant n$, is called the *elementary* subgroup of G. In case of the field, it coincides with the special linear group $\mathrm{SL}(n,K)$.

Similarly, by $d_i(\varepsilon) = e+(\varepsilon-1)e_{ii}$ we denote an *elementary pseudo-reflection*. For a given i we consider the corresponding **1-torus**

$$Q_i = \{d_i(\varepsilon),\ \varepsilon \in K^*\}.$$

Clearly, $\mathrm{GL}(n,K)$ is generated by $E(n,K)$ and Q_1.

Let $g \in G$. The largest subspace $W \leqslant V$, such that $g|_W = \mathrm{id}$ is called the *axis* of g. Similarly, the subspace $U = \{gv - v : \ v \in K^n\}$ is called the *centre* of g. Clearly, $\dim U = m$ and $\dim W = n - m$.

The elementary 2-torus $Q = Q_{U_0,W_0} = \{\mathrm{diag}(\varepsilon, \varepsilon, 1, \dots, 1),\ \varepsilon \in K^*\}$ is defined by the subspaces $U_0 = \langle e_1, e_2\rangle$ and $W_0 = \langle f_1, f_2\rangle$. It means, that elements of it are

$$d_0(\varepsilon) = e + e_1(\varepsilon - 1)f_1 + e_2(\varepsilon - 1)f_2, \quad \varepsilon \in K^*.$$

It is clear that

$$gQ_{UW}g^{-1} = Q_{gU,Wg^{-1}}, \quad g \in \mathrm{GL}(n, K).$$

Therefore any 2-torus (see [2]) is conjugated to the elementary 2-torus Q. The elements of an arbitrary 2-torus are the elements of the following form

$$d(\varepsilon) = e + u_1(\varepsilon - 1)v_1 + u_2(\varepsilon - 1)v_2, \quad \varepsilon \in K^*,$$

where $u_i = ge_i$, $v_i = f_i g^{-1}$, $1 \leqslant i \leqslant 2$, for some matrix $g \in \mathrm{GL}(n, K)$. Thus each 2-torus is completely determined by the subspaces $U = \langle u_1, u_2\rangle$ and $W = \langle v_1, v_2\rangle$.

The subspace U is precisely the *centre* of Q_{UW}, in the sense of being the centre of every $d(\varepsilon) \in Q_{UW}$, $\varepsilon \neq 1$. Similarly, the subspace $W^{\perp}$ orthogonal to $W \leqslant {}^nK$ with respect to the canonical pairing ${}^nK \times K^n \longrightarrow K$, is precisely the *axis* of Q_{UW}, in the above sense. Oftentimes we loosely refer to W itself as the axis of Q_{UW}.

Consider a pair of 2-tori X and Y with centers U_1 and U_2 and with axes W_1 and W_2, respictively. In paper [2] we introduce the following invariants for a pair of m-tori. $r = r(X, Y) = \dim(U_1 + U_2)$, $s = s(X, Y) = \dim(W_1 + W_2)$, $p = p(X, Y) = \dim(U_1 \cap W_2^{\perp})$, $q = q(X, Y) = \dim(U_2 \cap W_1^{\perp})$. Clearly that in the case of orbits in $\mathrm{GL}(4, K)$ we have $2 \leqslant r, s \leqslant 4$, $0 \leqslant p, q \leqslant 2$.

If a pair of 2-tori (X, Y) in $\mathrm{GL}(4, K)$ has invariants $r = s = 4$ we refer to it as *undegenerate* case and if at least one of invariants r or s less than 4 we refer to this pair as *degenerate* case. In this paper we calculate spans of degenerate cases. In the last work we treat with undegenerate cases.

3. Degenerate Cases

Let X, Y be 2-tori in $\mathrm{GL}(4, K)$ with centers U_1, U_2 and axes W_1, W_2, respectively. Choose some bases in these subspaces, $U_1 = \langle u_1, u_2\rangle$, $U_2 = \langle u_3, u_4\rangle$. $W_1 = \langle w_1, w_2\rangle$, $W_2 = \langle w_3, w_4\rangle$.

In the previous work [5] we described all orbits (X, Y) under action by simultaneous conjugation: (gXg^{-1}, gYg^{-1}), $g \in G$. For this purpose we listed all bases (u_1, u_2, u_3, u_4) and (w_1, w_2, w_3, w_4) corresponding to different orbits. Summarize these results in the next lemma.

Lemma 1. *Let X and Y be 2-tori in $\mathrm{GL}(4, K)$. Assume that $r = 2, 3$ and $r \leqslant s$, then the orbit (X, Y) is determined by one of the following bases.*

For $r = 2$, $s = 2$,

$$\begin{aligned} &u_1 = e_1,\ u_2 = e_2,\ w_1 = f_1,\ w_2 = f_2,\\ &u_3 = e_1,\ u_4 = e_2,\ w_3 = f_1,\ w_4 = f_2. \end{aligned} \tag{r2s2a}$$

For $r = 2$, $s = 3$,

$$\begin{aligned} &u_1 = e_1,\ u_2 = e_2,\ w_1 = f_1 + f_3,\ w_2 = f_2 + \lambda f_3,\\ &u_3 = e_1,\ u_4 = e_2,\ w_3 = f_1,\ w_4 = f_2, \end{aligned} \tag{r2s3a}$$

where $\lambda = 0$ or 1.

For $r = 2$, $s = 4$,

$$\begin{gathered} u_1 = e_1,\ u_2 = e_2,\ w_1 = f_1 + f_3,\ w_2 = f_2 + f_4, \\ u_3 = e_1,\ u_4 = e_2,\ w_3 = f_1,\ w_4 = f_2. \end{gathered} \tag{r2s4a}$$

For $r = 3$, $s = 3$,

$$\begin{gathered} u_1 = e_1,\ u_2 = e_2,\ w_1 = f_1 + f_4,\ w_2 = f_2 + \beta f_4, \\ u_3 = e_1 + e_3,\ u_4 = e_2 + \lambda e_3,\ w_3 = f_1,\ w_4 = f_2, \end{gathered} \tag{r3s3a}$$

where $\beta \in K$, $\lambda = 0$ *or 1.*

$$\begin{gathered} u_1 = e_1,\ u_2 = e_2,\ w_1 = f_1 + \alpha f_3,\ w_2 = f_2 + \beta f_3, \\ u_3 = e_1 + e_3,\ u_4 = e_2 + \lambda e_3,\ w_3 = f_1,\ w_4 = f_2, \end{gathered} \tag{r3s3b}$$

where $\lambda = 0$ *or 1,* $\alpha \in K^*$, $\beta \in K$.

For $r = 3$, $s = 4$, $p = 0$,

$$\begin{gathered} u_1 = e_1,\ u_2 = e_2,\ w_1 = f_1 + \alpha f_3 + f_4,\ w_2 = f_2 + \beta f_3, \\ u_3 = e_1 + e_3,\ u_4 = e_2 + \lambda e_3,\ w_3 = f_1,\ w_4 = f_2, \end{gathered} \tag{r3s4a}$$

where $\lambda = 0$ *or 1,* $\alpha \in K$, $\beta \in K^*$.

For $r = 3$, $s = 4$, $p = q = 1$,

$$\begin{gathered} u_1 = e_1,\ u_2 = e_2,\ w_1 = f_1 + \lambda_1 f_3 + f_4,\ w_2 = f_2 + \lambda_1\lambda_2 f_3, \\ u_3 = e_1 + \lambda_2 e_2,\ u_4 = e_3,\ w_3 = f_1,\ w_4 = f_3, \end{gathered} \tag{p1q1a}$$

where $\lambda_{1,2} = 0$ *or 1.*

In fact there are other orbits. They are obtained in two ways from listed bases. The first one is to consider a pair (Y, X) instead of (X, Y). These bases (orbits) we denoted in [5] adding prime $'$. It is clear that the span $\langle Y, X\rangle$ is not changed. The second way is to interchange rows and columns. In this case each set of bases with invariants (r, s), $r < s$, corresponds to a new set of bases with $r > s$. Then the new span $\langle X, Y\rangle$ is obtained from old one by matrix transposition.

4. Calculation Spans

In this section we prove our main results. They are listed in tables 1–3. Note that if the field $K = \mathbb{F}_2$ a torus coincides with the identity matrix. Thus this case is excluded.

For the description of spans we use new notations throughout the rest of the paper. $X^{\gamma_1}_{ij,\,km} = \{x_{ij}(\gamma_1\varepsilon)x_{km}(\varepsilon),\ \varepsilon \in K\}$, $X^{\gamma_1,\gamma_2}_{ij,\,km,\,pq} = \{x_{ij}(\gamma_1\varepsilon)x_{km}(\gamma_2\varepsilon)x_{pq}(\varepsilon),\ \varepsilon \in K\}$, $Q_{ij} = \{d_i(\varepsilon)d_j(\varepsilon),\ \varepsilon \in K\}$, where $\gamma_t \in K^*$, $t = 1, 2$. Note that we miss $\gamma_1 = 1$.

In all calculations we have deal with the generators of 2-tori $X = \langle x(\varepsilon),\ \varepsilon \in K\rangle$ and $Y = \langle y(\eta),\ \eta \in K\rangle$. The generators $x(\varepsilon)$ and $y(\eta)$ are constructed with the help of bases listed in Lemma 1. Sometimes we consider the generators conjugated to them. By H we denote the subgroup generated by $x(\varepsilon)$ and $y(\eta)$ $H = \langle x(\varepsilon), y(\eta),\ \varepsilon, \eta \in K^*\rangle$.

As usually $[g_1, g_2] = g_1 g_2 g_1^{-1} g_2^{-1}$ denotes a commutator of two elements g_1 and g_2. Also put $z(\varepsilon, \eta) = [x(\varepsilon), y(\eta)]$, $z_x(\varepsilon, \eta, \theta) = [z(\varepsilon, \eta), x(\theta)]$, $z_y(\varepsilon, \eta, \theta) = [z(\varepsilon, \eta), y(\theta)]$.

Now we are ready to formulate and prove our results. Note we do not write down the spans corresponding to cases with $r > s$. The reader can do it himself using remark at the end of the previous section.

Theorem 1. *Let* X*,* Y *be a pair of 2-tori in* $\mathrm{GL}(4, K)$*,* $K \neq \mathbb{F}_2$*. Suppose that* $r = 2$ *and* $r \leqslant s$*, then up to simultaneous conjugation* X *and* Y *generate one of the following subgroup* H*, listed in Table 1.*

Table 1: For $r = 2$.

	base	H
1	(r2s2a)	Q_{12}
2	(r2s3a), $\lambda = 1$	$Q_{12}X_{13,23}$
3	(r2s3a), $\lambda = 0$	$Q_{12}X_{13}$
4	(r2s4a)	$Q_{12}X_{13,24}$

$\lhd$ (1) For the base (r2s2a), we obtain directly that $H = Q_{12}$.

(2) For the base (r2s3a), the generators have the following form:

$$x(\varepsilon) = d_1(\varepsilon)d_2(\varepsilon)x_{13}\left(\frac{\varepsilon - 1}{\varepsilon}\right)x_{23}\left(\frac{\lambda(\varepsilon - 1)}{\varepsilon}\right), \quad y(\eta) = d_1(\eta)d_2(\eta).$$

Straightforward calculation shows that $z(\varepsilon, \eta) = x_{13}(-\theta_0)x_{23}(-\lambda\theta_0)$, where $\theta_0 = (\varepsilon-1)(\eta-1)$. Then from the decomposition of $x(\varepsilon)$ and $y(\eta)$ we get that $Q_{12}, X_{23,13}^{\lambda} \leqslant H$. Finally, we conclude that if $\lambda = 1$, $H = Q_{12}X_{13,23}$.

(3) For the base (r2s3a), if $\lambda = 0$, due to the previous argument, we obtain that $H = Q_{12}X_{13}$.

(4) For the base (r2s4a), by calculating $z(\varepsilon, \eta)$, we directly deduce that the subgroups Q_{12} and $X_{13,24}$ are contained in H. Thus we conclude that $H = Q_{12}X_{13,24}$. $\rhd$

Theorem 2. *Let* X*,* Y *be a pair of 2-tori in* $\mathrm{GL}(4, K)$*,* $K \neq \mathbb{F}_2$*. Suppose that* $r = s = 3$*, then up to simultaneous conjugation* X *and* Y *generate one of the following subgroup* H*, listed in Table 2.*

Table 2: For $r = 3$, $s = 3$.

	base	H
1	(r3s3a), $\lambda = 0$	$Q_{23}X_{14}X_{34,24}^{\beta}X_{12}$
2	(r3s3a), $\lambda = 1$	$Q_{23}X_{14}X_{34,24}^{\beta}X_{12,13}$
3	(r3s3b) $\alpha = -1$, $\lambda = 1$, $\beta = -1$, $\mathrm{Char}\, K = 2$ (r3s3b), $\alpha = -1$, $\lambda = 1$, $\beta = 1$ (r3s3b), $\alpha = 1$, $\lambda = 1$, $\beta = -1$ (r3s3b), $\alpha \neq 0, -1$, $\lambda = 1$, $\beta \neq 0, -1$, $1 + \alpha + \beta \neq 0$, $\alpha + \beta = 0$	$Q_{13}X_{12}X_{13}X_{23}$
4	(r3s3b), $\alpha = -1$, $\lambda = 1$, $\beta = -1$, $\mathrm{Char}\, K \neq 2$ (r3s3b), $\alpha = -1$, $\lambda = 1$, $\beta \neq -1, 0, 1$ (r3s3b), $\alpha \neq 0, -1$; $\lambda = 0$, $\beta \neq 0$ or $\lambda = 1$, $\beta = 0$ (r3s3b), $\alpha \neq 0, -1$, $\lambda = 0$, $\beta = 0$ (r3s3b), $\alpha \neq -1, 0, 1$, $\lambda = 1$, $\beta = -1$ (r3s3b), $\alpha \neq 0, -1$, $\lambda = 1$, $\beta \neq 0, -1$, $1 + \alpha + \beta \neq 0$, $\alpha + \beta \neq 0$	$\mathrm{GL}(2, K)$
5	(r3s3b), $\alpha = -1$, $\lambda = \beta = 0$ (r3s3b), $\alpha = -1$; $\lambda = 0$, $\beta \neq 0$ or $\lambda = 1$, $\beta = 0$ (r3s3b), $\alpha \neq 0, -1$, $\lambda = 1$, $\beta \neq 0, -1$, $1 + \alpha + \beta = 0$	$Q_{13}Q_{23}X_{12}$

$\lhd$ (1) For the base (r3s3a), a simultaneous conjugation by the element $\omega_{12}\omega_{23}$ leads to the following generators:

$$x(\varepsilon) = d_2(\varepsilon)d_3(\varepsilon)x_{24}\left(\frac{\varepsilon - 1}{\varepsilon}\right)x_{34}\left(\frac{\beta(\varepsilon - 1)}{\varepsilon}\right), \quad y(\eta) = d_2(\eta)d_3(\eta)x_{12}(\eta - 1)x_{13}(\lambda(\eta - 1)).$$

Straightforward calculation shows that

$$z(\varepsilon,\eta)=x_{12}\left(\frac{-\theta_0}{\varepsilon\eta}\right)x_{13}\left(\frac{-\lambda\theta_0}{\varepsilon\eta}\right)x_{14}\left(\frac{-(1+\beta\lambda)\theta_0}{\varepsilon}\right)x_{24}(-\theta_0)x_{34}(-\beta\theta_0),$$

where $\theta_0=(\varepsilon-1)(\eta-1)$. Calculate a commutator subgroup generated by $z(\varepsilon,\eta)$, we get that $X_{14}\leqslant H$.

Suppose that $\operatorname{Char} K\neq 2$, then

$$z(\varepsilon,\eta)z\left(\varepsilon,\frac{\eta}{2\eta-1}\right)=x_{14}(*)x_{24}\left(\frac{-2(\varepsilon-1)(\eta-1)^2}{2\eta-1}\right)x_{34}\left(\frac{-2\beta(\varepsilon-1)(\eta-1)^2}{2\eta-1}\right),$$

$$z(\varepsilon,\eta)z(\varepsilon,2-\eta)=x_{12}\left(\frac{-2(\varepsilon-1)(\eta-1)^2}{\varepsilon\eta(\eta-2)}\right)x_{13}\left(\frac{-2\lambda(\varepsilon-1)(\eta-1)^2}{\varepsilon\eta(\eta-2)}\right)x_{14}(*).$$

Suppose that $\operatorname{Char} K=2$, then

$$z(\varepsilon,\eta)z(\varepsilon,\varepsilon)z(\varepsilon,1+\varepsilon+\eta)=x_{14}(*)x_{12}\left(\frac{\theta_1}{\varepsilon^2\eta(1+\varepsilon+\eta)}\right)x_{13}\left(\frac{\lambda\theta_1}{\varepsilon^2\eta(1+\varepsilon+\eta)}\right),$$

$$z(\varepsilon,\eta)z(\varepsilon,\varepsilon)z\left(\varepsilon,\frac{\varepsilon\eta}{\varepsilon\eta+\varepsilon+\eta}\right)=x_{14}(*)x_{24}\left(\frac{\theta_1}{\varepsilon(\eta+1)+\eta}\right)x_{34}\left(\frac{\beta\theta_1}{\varepsilon(\eta+1)+\eta}\right),$$

where $\theta_1=(\varepsilon+1)\left(\eta(\eta+1)+\varepsilon^2(\eta+1)+\varepsilon(\eta^2+1)\right)$. In all cases the subgroups $X^{\beta}_{34,24}$, $X^{\lambda}_{13,12}$ and X_{14} are contained in H. Thus we can conclude that if $\lambda=0$, $H=Q_{23}X_{14}X^{\beta}_{34,24}X_{12}$.

(2) For the base (r3s3a), if $\lambda=1$, due to the previous argument, we obtain that $H=Q_{23}X_{14}X^{\beta}_{34,24}X_{12,13}$.

(3) For the base (r3s3b), $\alpha=-1$, $\lambda=1$, $\beta=-1$. Simultaneous conjugation by $w_{23}w_{12}x_{12}(1)$ yields the following generators:

$$x(\varepsilon)=d_1(\varepsilon)d_3(\varepsilon)x_{12}\left(\frac{1-\varepsilon}{\varepsilon}\right)x_{32}\left(\frac{2(1-\varepsilon)}{\varepsilon}\right),\quad y(\eta)=d_1(\eta)d_3(\eta)x_{23}(\eta-1).$$

If $\operatorname{Char} K=2$, the group $\langle X,Y\rangle$ is embedded in a group of upper triangular matrices, and the generators have the following form:

$$x(\varepsilon)=d_1(\varepsilon)d_3(\varepsilon)x_{12}\left(\frac{1-\varepsilon}{\varepsilon}\right),\quad y(\eta)=d_1(\eta)d_3(\eta)x_{23}(\eta-1). \tag{A}$$

A direct calculation reveals that $z(\varepsilon,\eta)=x_{12}(\theta_2)x_{13}\left(\frac{\theta_2(1+\varepsilon\eta+\varepsilon)}{\varepsilon\eta}\right)x_{23}\left(\frac{\theta_2}{\varepsilon\eta}\right)$, where $\theta_2=(\varepsilon+1)(\eta+1)$. By calculating a commutator subgroup generated by $z(\varepsilon,\eta)$, we get that $[z(\varepsilon_1,\eta_1),z(\varepsilon_2,\eta_2)]$ coincides with the subgroup X_{13}.

Calculate the following products:

$$z(\varepsilon,\eta)z(\varepsilon,\varepsilon)z\left(\varepsilon,\frac{\varepsilon(\varepsilon+1)\eta}{\varepsilon+\varepsilon^2+\eta+\varepsilon^2\eta}\right)=x_{12}\left(\frac{(\varepsilon+1)(\eta+1)(\varepsilon+\eta)(\varepsilon^2+1)}{\varepsilon+\varepsilon^2\eta+\varepsilon^2+\eta}\right)x_{13}(*),$$

$$z(\varepsilon,\eta)z(\varepsilon,\varepsilon)z\left(\varepsilon,\frac{1+\varepsilon^2+\eta+\varepsilon\eta}{\varepsilon+1}\right)=x_{23}\left(\frac{(\varepsilon+1)(\eta+1)(\varepsilon+\eta)(\varepsilon^2+1)}{\varepsilon^2\eta(1+\varepsilon^2+\eta+\varepsilon\eta)}\right)x_{13}(*).$$

From the above, we derive that X_{12}, X_{13}, X_{23} are the subgroups of H. Thus $H=Q_{13}X_{12}X_{13}X_{23}$, if $\operatorname{Char} K=2$. We will consider the case of $\operatorname{Char} K\neq 2$ below.

• For the base (r3s3b), $\alpha = -1$, $\lambda = 1$, $\beta \neq 0, -1$. Let $\beta = 1$, the action of simultaneous conjugation by $w_{12}w_{13}x_{12}(1)$ results in the generators given below:

$$x(\varepsilon) = d_1(\varepsilon)d_3(\varepsilon)x_{12}\left(\frac{\varepsilon - 1}{\varepsilon}\right), \quad y(\eta) = d_1(\eta)d_3(\eta)x_{23}(\eta - 1).$$

Applying the similar arguments as to that for generators (A) yields that $H = Q_{13}X_{12}X_{13}X_{23}$.

• For the base (r3s3b), $\alpha = 1$, $\lambda = 1$, $\beta = -1$. Conjugating this base by the element $w_{12}w_{13}x_{12}(1)$, we have new generators

$$x(\varepsilon) = d_1(\varepsilon)d_3(\varepsilon)x_{12}\left(\frac{1 - \varepsilon}{\varepsilon}\right), \quad y(\eta) = d_1(\eta)d_3(\eta)x_{23}(\eta - 1).$$

Similar to the generators (A), we obtain that $H = Q_{13}X_{12}X_{13}X_{23}$.

• For the base (r3s3b), $\alpha \neq 0, -1$, $\lambda = 1$, $\beta \neq 0, -1$, $1 + \alpha + \beta \neq 0$. Let $\alpha + \beta = 0$. Conjugating the generators by the element $w_{23}w_{12}x_{21}(-1)x_{12}(1)$ produces the following new generators:

$$x(\varepsilon) = d_1(\varepsilon)d_3(\varepsilon)x_{12}\left(\frac{\alpha(1 - \varepsilon)}{\varepsilon}\right), \quad y(\eta) = d_1(\eta)d_3(\eta)x_{23}(\eta - 1).$$

An argument analogous to that for generators (A) shows that $H = Q_{13}X_{12}X_{13}X_{23}$.

(4) For the base (r3s3b), $\alpha = -1$, $\lambda = 1$, $\beta = -1$. Suppose that $\operatorname{Char} K \neq 2, 3$, conjugating the original base by the element $x_{12}(-\frac{1}{2})w_{12}x_{12}(1)$, we get the new generators

$$x(\varepsilon) = d_1(\varepsilon)d_2(\varepsilon)x_{23}\left(\frac{2(1 - \varepsilon)}{\varepsilon}\right), \quad y(\eta) = d_1(\eta)d_2(\eta)x_{32}(\eta - 1).$$

For $\varepsilon \neq 1$, $\eta \neq 1$, $z_x(\varepsilon, 1/2, \eta) = x_{23}(-3(\varepsilon - 1)\varepsilon(\eta - 1))$. We get that X_{23} is contained in H. It follows from the decomposition of $x(\varepsilon)$ and $y(\eta)$ that X_{32}, $Q_{12} \leqslant H$.

Define a map $\phi : Q_{12} \to \{\operatorname{diag}(a, 1) : a \in K^*\} \subset \mathrm{GL}(2, K)$ by $\phi \operatorname{diag}(a, a, 1, 1) = \operatorname{diag}(a, 1)$. It is easy to see that ϕ is a surjective and injective homomorphism. Moreover $\langle X_{32}, X_{23}\rangle \cong \mathrm{SL}(2, K)$. In fact, $\langle \{\operatorname{diag}(a, 1) : a \in K^*\}, \mathrm{SL}(2, K)\rangle \cong \mathrm{GL}(2, K)$, therefore $\langle X_{32}, Q_{12}, X_{23}\rangle \cong \mathrm{GL}(2, K)$. Finally we obtain that $H = \mathrm{GL}(2, K)$, if $\operatorname{Char} K \neq 2, 3$.

If $\operatorname{Char} K = 3$, the original generators become the following form:

$$x'(\varepsilon) = d_1(\varepsilon)d_2(\varepsilon)x_{13}\left(\frac{2(\varepsilon - 1)}{\varepsilon}\right)x_{23}\left(\frac{2(\varepsilon - 1)}{\varepsilon}\right),$$

$$y'(\eta) = d_1(\eta)d_2(\eta)x_{31}(\eta - 1)x_{32}(\eta - 1).$$

Conjugating this base by the element $x_{12}(1)w_{12}x_{12}(1)$ we get new generators

$$x(\varepsilon) = d_1(\varepsilon)d_2(\varepsilon)x_{23}\left(\frac{\varepsilon - 1}{\varepsilon}\right), \quad y(\eta) = d_1(\eta)d_2(\eta)x_{32}(\eta - 1).$$

Put $P(\varepsilon, \eta) = x(\varepsilon)y(\eta)x\left(\frac{2\varepsilon\eta + \varepsilon + 2\eta + 2}{2\varepsilon\eta + 2\varepsilon + 2\eta + 1}\right)$. For $2 + \varepsilon + \eta + \varepsilon\eta \neq 0$, $1 + 2\varepsilon + \eta + 2\varepsilon\eta \neq 0$, $\theta \neq 1$, we have

$$[P(\varepsilon, \eta), P(2, \theta)] = x_{32}\left(\frac{2\varepsilon^2(\eta^2 + 2)(2\theta + 1) + 2(\eta^2 + 2)(\theta + 2)}{(2\varepsilon\eta + \varepsilon + 2\eta + 2)(\varepsilon(2\eta + 2) + 2\eta + 1)}\right).$$

It follows that X_{32} is contained in H. Then we extract the subgroups X_{23} and Q_{12} from the decomposition of generators. Finally, we conclude that $H = \mathrm{GL}(2, K)$, when $\operatorname{Char} K = 3$.

• For the base (r3s3b), $\alpha = -1$, $\lambda = 1$, $\beta \neq 0, -1$. Let $\beta \neq 1$. Conjugate by $x_{23}\big(\frac{\beta}{1-\beta}\big)x_{32}\big(\frac{\beta-1}{\beta}\big)d_1(\beta)w_{13}x_{12}\big(\frac{1}{\beta}\big)$. Thus the generators become the following:

$$x(\varepsilon) = d_2(\varepsilon)d_3(\varepsilon)x_{31}\left(\frac{(\beta-1)(\varepsilon-1)}{\beta\varepsilon}\right), \quad y(\eta) = d_2(\eta)d_3(\eta)x_{13}(\beta(\eta-1)). \tag{B}$$

Now suppose that $\operatorname{Char} K \neq 2$. Let $\beta \neq \frac{1}{2}$. For $\varepsilon \neq 1$, $\eta \neq 1$, it follows directly from calculation that

$$z_x\left(\varepsilon, \frac{\beta}{\beta-1}, \eta\right) = x_{31}\left(\frac{\varepsilon(2\beta-1)(\varepsilon-1)(\eta-1)}{\beta}\right).$$

Let $\beta = \frac{1}{2}$, put

$$f(\varepsilon, \eta) = z_y\left(\varepsilon, \eta, \frac{1+\varepsilon-\eta+\varepsilon\eta}{-1+\varepsilon+\eta+\varepsilon\eta}\right)$$

$$= d_1\left(\frac{(\varepsilon\eta+\varepsilon-\eta+1)(\varepsilon\eta+\varepsilon+\eta-1)}{4\varepsilon^2\eta}\right) d_3\left(\frac{4\varepsilon^2\eta}{(\varepsilon\eta+\varepsilon-\eta+1)(\varepsilon\eta+\varepsilon+\eta-1)}\right)$$

$$\times x_{31}\left(\frac{\left(\varepsilon^2-1\right)(\eta-1)^2(\varepsilon\eta+\varepsilon-\eta+1)\left(\varepsilon^2\eta-\varepsilon^2+2\varepsilon\eta+2\varepsilon+\eta-1\right)}{16\varepsilon^4\eta^2}\right).$$

Then we have $[f(\varepsilon_1, \eta_1), f(\varepsilon_2, \eta_2)] = X_{31}$.

In the case of characteristic 2, $z_x(\varepsilon, \beta(\beta+1), \varepsilon) = x_{31}\left(\varepsilon(\varepsilon+1)^2/\beta\right)$.

Next, from the decomposition of the generators we get that the subgroups X_{13}, X_{31} and Q_{23} are contained in H. Similarly, we conclude that $H = \mathrm{GL}(2, K)$.

• For the base (r3s3b), $\alpha \neq 0, -1$, $\lambda = 0$, $\beta \neq 0$. A simultaneous conjugation by the element $w_{13}w_{12}x_{21}\big(\frac{\beta}{\alpha}\big)x_{12}\big(-\frac{\alpha}{\beta}\big)$ leads to the following generators:

$$x(\varepsilon) = d_2(\varepsilon)d_3(\varepsilon)x_{31}\left(\frac{\beta(\varepsilon-1)}{\varepsilon}\right), \quad y(\eta) = d_2(\eta)d_3(\eta)x_{13}\left(\frac{\alpha(\eta-1)}{\beta}\right).$$

An argument similar to the generators (B) gives that $H = \mathrm{GL}(2, K)$.

• For the base (r3s3b), $\alpha \neq 0, -1$, $\lambda = 1$, $\beta = 0$, conjugating by the element $x_{13}(1)w_{23}$, we can extract the subgroups Q_{13}, X_{12} and X_{21} and then $H = \mathrm{GL}(2, K)$.

• For the base (r3s3b), $\alpha \neq 0, -1$, $\lambda = 0$, $\beta = 0$, the generators have the following form:

$$x(\varepsilon) = d_1(\varepsilon)d_2(\varepsilon)x_{13}\left(\frac{\alpha(\varepsilon-1)}{\varepsilon}\right), \quad y(\eta) = d_1(\eta)d_2(\eta)x_{31}(\eta-1).$$

The similar calculations to the generators (B) yield $H = \mathrm{GL}(2, K)$.

• For the base (r3s3b), $\alpha \neq 0, -1, 1$, $\lambda = 1$, $\beta = -1$. Performing conjugation by $x_{13}\big(\frac{1}{\alpha-1}\big)x_{31}(1-\alpha)w_{23}w_{12}x_{12}(\alpha)$ gives rise to the generators listed below:

$$x(\varepsilon) = d_1(\varepsilon)d_3(\varepsilon)x_{32}\left(\frac{(\alpha-1)(\varepsilon-1)}{\varepsilon}\right), \quad y(\eta) = d_1(\eta)d_3(\eta)x_{23}(\eta-1).$$

Now suppose that $\operatorname{Char} K \neq 2$. Let $\alpha \neq \frac{1}{2}$, for the element $\theta \in K$, we have

$$z_x\left(\varepsilon, \frac{\alpha}{\alpha-1}, \theta\right) = x_{32}\left((2\alpha-1)\varepsilon(\varepsilon-1)(\theta-1)\right).$$

Let $\alpha = \frac{1}{2}$, after similar calculations to the case $\beta = \frac{1}{2}$ of the base (r3s3b), $\alpha = -1$, $\lambda = 1$, $\beta \neq 0, -1$, we have the subgroup X_{23}.

Assume that $\operatorname{Char} K = 2$, $z_x\big(\varepsilon, \frac{\alpha}{\alpha+1}, \varepsilon\big) = x_{32}\left(\varepsilon(\varepsilon+1)^2\right)$. Thus in all cases, it follows that X_{32}, Q_{13} and X_{23} are contained in H.

• For the base (r3s3b), $\alpha \neq 0, -1$, $\lambda = 1$, $\beta \neq 0, -1$, $1 + \alpha + \beta \neq 0$. Under condition $\alpha + \beta \neq 0$, conjugate this base by the element $x_{23}\big(-\frac{\beta}{\alpha+\beta}\big)x_{32}\big(\frac{\beta+\alpha}{\beta}\big)d_1(\beta)w_{13}x_{12}\big(-\frac{\alpha}{\beta}\big)$. We have new generators

$$x(\varepsilon) = d_2(\varepsilon)d_3(\varepsilon)x_{31}\left(\frac{(\alpha+\beta)(\varepsilon-1)}{\beta\varepsilon}\right), \quad y(\varepsilon) = d_2(\varepsilon)d_3(\varepsilon)x_{13}(\beta(\varepsilon-1)).$$

Similar to the generators (B), we can conclude that X_{13}, Q_{23} and X_{31} are subgroups of H.

(5) For the base (r3s3b), $\alpha = -1$, $\lambda = \beta = 0$. With the help of simultaneous conjugation by $w_{12}x_{12}(-1)w_{23}$, the generators of the group X and Y have the following form:

$$x(\varepsilon) = d_2(\varepsilon)d_3(\varepsilon), \quad y(\eta) = d_1(\eta)d_3(\eta)x_{12}\left(\frac{\eta-1}{\eta}\right).$$

We calculate directly that $z(\varepsilon,\eta) = x_{12}(-(\varepsilon-1)(\eta-1)/\varepsilon)$, it follows from the decomposition of the generators that the subgroups Q_{13}, X_{12} and Q_{23} are contained in H. Finally we can obtain that $H = Q_{13}Q_{23}X_{12}$.

• For the base (r3s3b), $\alpha = -1$, $\lambda = 0$, $\beta \neq 0$. Conjugate this base by the element $x_{13}(-\beta)w_{23}x_{23}(-1)x_{12}(\beta)w_{13}w_{23}$. Then we get a new base

$$x(\varepsilon) = d_1(\varepsilon)d_3(\varepsilon)x_{32}\left(\frac{(\alpha-1)(\varepsilon-1)}{\varepsilon}\right), \quad y(\eta) = d_1(\eta)d_3(\eta)x_{23}(\eta-1).$$

Then $z(\varepsilon,\eta) = x_{23}\left((\varepsilon-1)(\eta-1)/\varepsilon\right)$. We consecutively get that the subgroups Q_{12}, Q_{13} and X_{23} are contained in H. Then under conjugation of the element $w_{13}w_{23}$, we have that $H = Q_{13}Q_{23}X_{12}$.

• For the base (r3s3b), $\alpha = -1$, $\lambda = 1$, $\beta = 0$. Conjugate this base by the element $w_{12}x_{12}(-1)x_{13}(1)w_{23}$. Then we get the same generators as in the base (r3s3b), $\alpha = -1$, $\lambda = \beta = 0$.

• For the base (r3s3b), $\alpha \neq 0, -1$, $\lambda = 1$, $\beta \neq 0, -1$, $1+\alpha+\beta = 0$. Simultaneous conjugation by $x_{13}(-1-\alpha)x_{21}(1)x_{13}(1+\alpha)d_1(-1-\alpha))w_{13}x_{12}\big(\frac{\alpha}{1+\alpha}\big)$ yields the following generators:

$$x(\varepsilon) = d_2(\varepsilon)d_3(\varepsilon), \quad y(\eta) = d_1(\eta)d_3(\varepsilon)x_{12}\left(\frac{1-\eta}{\eta}\right).$$

Direct calculation shows that $z(\varepsilon,\eta) = x_{12}\left((\varepsilon-1)(\eta-1)/\varepsilon\right)$. We consecutively get that the subgroups Q_{13}, Q_{23} and X_{12} are contained in H. Finally we can conclude that $H = Q_{13}Q_{23}X_{12}$. ▷

Theorem 3. *Let X, Y be a pair of 2-tori in $\mathrm{GL}(4, K)$, $K \neq \mathbb{F}_2$. Suppose that $r = 3$, $s = 4$, then up to simultaneous conjugation X and Y generate one of the following subgroup H, listed in Table 3. In cases (4) and (5) below, we also suppose that $K \neq \mathbb{F}_3$.*

Table 3: For $r = 3$, $s = 4$.

	base	H
1	(r3s4a), $\alpha = -1$, $\lambda = 0$, $\beta \neq 0$	$XYX_{14}X^{\beta,-1}_{13,23,24}$
2	(r3s4a), $\alpha = 0$, $\lambda = 1$, $\beta = -1$	$Q_{12}Q_{23}X_{13}X_{24}$
3	(r3s4a), $\alpha \neq 0, -1$, $\lambda = 1$, $\beta \neq 0, -1$, $1 + \alpha + \beta = 0$	$XYX_{14}X^{1+\alpha}_{13,23,24}$
4	(r3s4a), $\alpha = 0$, $\lambda = 0$, $\beta \neq 0$ (r3s4a), $\alpha = -1$, $\lambda = 1$, $\beta = -1$, $\mathrm{Char}\, K = 2$ (r3s4a), $\alpha = -1$, $\lambda = 1$, $\beta = 1$ (r3s4a), $\alpha = 1$, $\lambda = 1$, $\beta = -1$ (r3s4a), $\alpha \neq 0, -1$, $\lambda = 1$, $\beta \neq 0, -1$, $1 + \alpha + \beta \neq 0$, $\alpha + \beta = 0$	$Q_{13}X^{\beta}_{12,34}X_{23}X^{-\beta}_{13,24}X_{14}$
5	(r3s4a), $\alpha \neq 0, -1$, $\lambda = 0$, $\beta = -1$ (r3s4a), $\alpha = 0$, $\lambda = 1$, $\beta \neq 0, -1$ (r3s4a), $\alpha \neq 0, -1, 1$, $\lambda = 1$, $\beta = -1$ (r3s4a), $\alpha \neq 0, -1$, $\lambda = 1$, $\beta \neq 0, -1$, $1 + \alpha + \beta \neq 0$, $\alpha + \beta \neq 0$	$\mathrm{GL}(2, K)X_{24}$
6	(r3s4a), $\alpha = -1$, $\lambda = 1$, $\beta = -1$, $\mathrm{Char}\, K \neq 2$	$\mathrm{GL}(2, K)X_{14}$
7	(r3s4a), $\alpha = -1$, $\lambda = 1$, $\beta \neq 0, -1, 1$	$\mathrm{GL}(2, K)X^{\frac{\beta}{1-\beta}}_{24,34}$
8	(p1q1a)	$Q_{12}Q_{23}X_{24}$

$\lhd$ (1) For the base (r3s4a), $\alpha = -1$, $\lambda = 0$, $\beta \neq 0$. Conjugate this base by the element $w_{12}x_{31}(-1)$. Then the generators of the group X and Y have the following form:

$$x(\varepsilon) = d_1(\varepsilon)d_3(\varepsilon)x_{12}\left(\frac{\beta(\varepsilon-1)}{\varepsilon}\right)x_{13}\left(\frac{\beta(\varepsilon-1)}{\varepsilon}\right)x_{23}(1-\varepsilon)x_{24}(\varepsilon-1)x_{34}\left(\frac{1-\varepsilon}{\varepsilon}\right),$$
$$y(\eta) = d_1(\eta)d_2(\eta).$$

For $\varepsilon, \eta \neq 1$, we have

$$z(\varepsilon,\eta) = x_{13}\left(\frac{-\beta\theta_0}{\varepsilon}\right)x_{14}\left(\frac{-\beta\theta_0(\varepsilon-1)}{\varepsilon}\right)x_{23}\left(\frac{\theta_0}{\varepsilon}\right)x_{24}\left(\frac{-\theta_0}{\varepsilon}\right),$$

where $\theta_0 = (\varepsilon-1)(\eta-1)$. We find that $z_x(\varepsilon, \eta, -1/\varepsilon) = x_{13}(\beta\theta_3)x_{23}(-\theta_3)x_{24}(\theta_3)$, where $\theta_3 = -(\varepsilon^2-1)(\eta-1)/\varepsilon$. It follows that $X^{\beta,-1}_{13,23,24} \leqslant H$. After it we get X_{14}. Since Y commutes with $X^{\beta,-1}_{13,23,24}$, we conclude that $H = XYX_{14}X^{\beta,-1}_{13,23,24}$.

(2) For (r3s4a), $\lambda = 1$, $\alpha = 0$, $\beta = -1$. The generators obtaining after conjugation by $x_{24}(-1)x_{34}(1)x_{23}(-1)x_{14}(-1)x_{32}(1)x_{21}(-1)w_{13}$ are as follows:

$$x(\varepsilon) = d_2(\varepsilon)d_3(\varepsilon), \quad y(\eta) = d_1(\eta)d_2(\eta)x_{13}\left(\frac{\eta-1}{\eta}\right)x_{24}\left(\frac{\eta-1}{\eta}\right).$$

For $\varepsilon, \eta \neq 1$, we have $z(\varepsilon,\eta) = x_{13}\left(-(\varepsilon-1)(\eta-1)/\varepsilon\right)x_{24}\left((\varepsilon-1)(\eta-1)\right)$. Put $f_1(\varepsilon,\eta) = z(\varepsilon,\eta)y\left((\varepsilon\eta-\eta+1)/\varepsilon\right)$. We have that $[f_1(\varepsilon_1,\eta_1), f_1(\varepsilon_2,\eta_2)] = X_{24}$. Next we get X_{13}. Finally, we conclude that $H = Q_{12}Q_{23}X_{13}X_{24}$.

(3) For the base (r3s4a), $\lambda = 1$, $\alpha \neq 0, -1$, $\beta \neq 0, -1$, $1 + \alpha + \beta = 0$. Conjugate this base by the element $w_{23}x_{23}(-1)w_{12}x_{12}(1)$. Then the generators of the group X and Y have the following form:

$$x(\varepsilon) = d_1(\varepsilon)d_3(\varepsilon)x_{12}\left(-\frac{(1+\alpha)(\varepsilon-1)}{\varepsilon}\right)x_{34}\left(\frac{\varepsilon-1}{\varepsilon}\right), \quad y(\eta) = d_1(\eta)d_2(\eta)x_{23}\left(\frac{\eta-1}{\eta}\right).$$

Simple calculations show that

$$z(\varepsilon,\eta) = x_{13}\left(\frac{-(1+\alpha)\theta_0}{\varepsilon}\right)x_{14}\left(\frac{(1+\alpha)(\varepsilon-1)\theta_0}{\varepsilon}\right)x_{23}\left(\frac{-\theta_0}{\varepsilon}\right)x_{24}\left(\frac{-\theta_0}{\varepsilon}\right),$$

where $\theta_0 = (\varepsilon - 1)(\eta - 1)$. Moreover, we put $f_1(\varepsilon, \eta) = z(\varepsilon, \eta) y\left((\varepsilon\eta - \eta + 1)/\varepsilon\right)$. And $[f_1(\varepsilon_1, \eta_1), f_1(\varepsilon_2, \eta_2)]$ equals X_{14}. Next we get $X^{1+\alpha}_{13,23,24}$. Finally, we conclude that $H = XYX_{14}X^{1+\alpha}_{13,23,24}$.

(4) For the base (r3s4a), $\alpha = 0$, $\lambda = 0$, $\beta \neq 0$. Simultaneous conjugation by $w_{23}w_{12}$ leads to the following generators:

$$x(\varepsilon) = d_1(\varepsilon)d_3(\varepsilon)x_{12}\left(\frac{\beta(\varepsilon-1)}{\varepsilon}\right)x_{34}\left(\frac{\varepsilon-1}{\varepsilon}\right), \quad y(\eta) = d_1(\eta)d_3(\eta)x_{23}(\eta-1).$$

Performing the straightforward calculation, we obtain that $z(\varepsilon, \eta)$ equals

$$x_{12}(-\beta\theta_0)x_{13}\left(\frac{\beta\theta_0(1+\varepsilon(\eta-1))}{\varepsilon\eta}\right)x_{14}\left(\frac{-\beta(\varepsilon-1)\theta_0}{\varepsilon}\right)x_{23}\left(\frac{-\theta_0}{\varepsilon\eta}\right)x_{24}\left(\frac{-\theta_0}{\varepsilon}\right)x_{34}(-\theta_0),$$

where $\theta_0 = (\varepsilon - 1)(\eta - 1)$.

Next, suppose that $\operatorname{Char} K \neq 2$,

$$[z(\varepsilon,\eta), z(\varepsilon, 3-\eta)] = x_{13}(\beta\theta_4)x_{24}(-\theta_4), \quad [z(\varepsilon,\eta), z(\eta,\varepsilon)] = x_{14}(\beta\theta_5),$$

where

$$\theta_4 = \frac{(\varepsilon-1)^2\left(2\eta^3 - 9\eta^2 + 13\eta - 6\right)}{\varepsilon(\eta-3)\eta}, \quad \theta_5 = \frac{2(\varepsilon-1)^2(\eta-1)^2(\varepsilon-\eta)}{\varepsilon\eta}.$$

Take $\varepsilon \neq 1$, $\eta \neq 1, \frac{1}{2}, 2, 3$, $\varepsilon \neq \eta$. Hence for some $\theta \in K^*$, $x_{13}(-\beta\theta)x_{24}(\theta)$ and $x_{14}(\theta)$ lie in H.

On the other hand, for $\eta \neq 1, 2, \frac{1}{2}$,

$$z(\varepsilon,\eta)z\left(\varepsilon, \frac{\eta}{2\eta-1}\right) = x_{12}\left(\frac{-2\beta(\eta-1)\theta_0}{2\eta-1}\right)x_{34}\left(\frac{-2(\eta-1)\theta_0}{2\eta-1}\right)$$
$$\times\ x_{13}(-\beta q_1(\varepsilon,\eta))x_{24}(q_1(\varepsilon,\eta))x_{14}(q_2(\varepsilon,\eta)),$$

$$z(\varepsilon,\eta)z\left(\varepsilon, 2-\eta\right) = x_{13}\left(\frac{\beta\theta_0^2}{\varepsilon\eta}\right)x_{24}\left(\frac{-\theta_0^2}{\varepsilon\eta}\right)x_{14}\left(\frac{\beta\theta_0^3}{\varepsilon\eta}\right)x_{23}\left(\frac{-2(\eta-1)\theta_0}{\varepsilon\eta(\eta-2)}\right),$$

where $q_1(\varepsilon,\eta)$ and $q_2(\varepsilon,\eta)$ are rational functions which have pole at point $\eta = \frac{1}{2}$. But we know that $X^{-\beta}_{13,24}$ and X_{14} are contained in H. Hence we can extract the elements of the groups $X^{\beta}_{12,34}$ and X_{23}, after it we get the whole groups.

Let $\operatorname{Char} K = 2$. Then

$$[z(\varepsilon,\eta), z(\varepsilon, 1+\eta)] = x_{13}\left(-\frac{\beta(\varepsilon+1)\theta_0}{\varepsilon(\eta+1)}\right)x_{24}\left(\frac{(\varepsilon+1)\theta_0}{\varepsilon(\eta+1)}\right),$$

$$\left[z(\varepsilon,\eta), z\left(\varepsilon+1, \frac{\varepsilon\eta}{\varepsilon+1}\right)\right] = x_{14}\left(\frac{\beta\theta_0\left(\varepsilon^2(\eta+1) + \varepsilon\eta^2 + \eta + 1\right)}{(\varepsilon+1)^2\eta}\right).$$

One readily calculates that

$$z(\varepsilon,\eta)z\left(\varepsilon+1, \frac{\varepsilon^2\eta}{\varepsilon^2+\eta+1}\right) = x_{12}(\beta\theta_6)x_{34}(\theta_6)x_{13}(-\beta g_1(\varepsilon,\eta))x_{24}(g_1(\varepsilon,\eta))x_{14}(g_2(\varepsilon,\eta)),$$

where $\theta_6 = \frac{(\eta+1)(\varepsilon+1)(\varepsilon+\eta+1)}{\varepsilon^2+\eta+1}$, $g_1(\varepsilon,\eta)$ and $g_2(\varepsilon,\eta)$ are rational functions which have poles at points $\varepsilon = -1$, $\eta = 1+\varepsilon^2$. Take arbitrary ε and η except for $\varepsilon+\eta+1 = 0$, $\varepsilon = 1$ and $\eta = 1$, we get all elements of the group $X^{\beta}_{12,34}$.

For $\varepsilon \neq 1$, $\varepsilon\eta+\eta+1\neq 0$, multiplying $z(\varepsilon,\eta)z\left(\varepsilon+1, \frac{\varepsilon\eta+\eta+1}{\varepsilon}\right)$ by suitable elements from $X_{13,24}^{-\beta}$ and X_{14}, we can get an element $x_{23}\left(\frac{(\eta+1)(\varepsilon+\eta+1)}{\varepsilon\eta(\varepsilon\eta+\eta+1)}\right)$. Take arbitrary ε and η except for $\varepsilon+\eta+1=0$ and $\eta=1$, we get the subgroup X_{23}. In all cases we get the subgroups $X_{12,34}^{\beta}$, X_{23}, $X_{13,24}^{-\beta}$, X_{14}. Finally, we conclude that $H = Q_{13}X_{12,34}^{\beta}X_{23}X_{13,24}^{-\beta}X_{14}$.

• For the base (r3s4a), $\alpha=-1$, $\lambda=1$, $\beta=-1$. If $\operatorname{Char} K=2$, conjugating this base by the element $w_{23}w_{12}x_{12}(1)$, the group $\langle X,Y\rangle$ is embedded in a group of upper triangular matrices, and the generators have the following form:

$$x(\varepsilon)=d_1(\varepsilon)d_3(\varepsilon)x_{12}\left(\frac{1-\varepsilon}{\varepsilon}\right)x_{34}\left(\frac{\varepsilon-1}{\varepsilon}\right),\quad y(\eta)=d_1(\eta)d_3(\eta)x_{23}(\eta-1).$$

The generators are special case of the base (r3s4a), $\alpha=0$, $\lambda=0$, $\beta\neq 0$. Let $\beta=-1$, we obtain that $H=Q_{13}X_{12,34}^{-1}X_{23}X_{13,24}X_{14}$, if $\operatorname{Char} K=2$. We will consider the case of $\operatorname{Char} K\neq 2$ below.

• For the base (r3s4a), $\alpha=-1$, $\lambda=1$, $\beta\neq 0,-1$. Let $\beta=1$. Conjugation by the element $w_{23}w_{12}x_{12}(1)$ leads to the following generators:

$$x(\varepsilon)=d_1(\varepsilon)d_3(\varepsilon)x_{12}\left(\frac{\varepsilon-1}{\varepsilon}\right)x_{34}\left(\frac{\varepsilon-1}{\varepsilon}\right),\quad y(\eta)=d_1(\eta)d_3(\eta)x_{23}(\eta-1).$$

The generators are special case of the base (r3s4a), $\alpha=0$, $\lambda=0$, $\beta=1$, we have that $H=Q_{13}X_{12,34}X_{23}X_{13,24}^{-1}X_{14}$.

• For the base (r3s4a), $\alpha\neq 0,-1$, $\lambda=1$, $\beta=-1$. If $\alpha=1$, conjugating this base by the element $w_{23}w_{12}x_{12}(1)$, the group $\langle X,Y\rangle$ is embedded in a group of upper triangular matrices, and the generators are special case of the base (r3s4a), $\alpha=0$, $\lambda=0$, $\beta\neq 0$. Let $\beta=-1$, we obtain that $H=Q_{13}X_{13,24}X_{14}X_{23}X_{12,34}^{-1}$.

• For the base (r3s4a), $\alpha\neq 0,-1$, $\lambda=1$, $\beta\neq 0,-1$, $1+\alpha+\beta\neq 0$. If $\alpha+\beta=0$, conjugating by $w_{23}w_{12}x_{21}(-1)x_{12}(1)$, the generators become the following:

$$x(\varepsilon)=d_1(\varepsilon)d_3(\varepsilon)x_{12}\left(\frac{\alpha(1-\varepsilon)}{\varepsilon}\right)x_{14}\left(\frac{1-\varepsilon}{\varepsilon}\right)x_{24}\left(\frac{\varepsilon-1}{\varepsilon}\right),\quad y(\eta)=d_1(\eta)d_3(\eta)x_{23}(\eta-1).$$

Direct calculation yields that $z(\varepsilon,\eta)$ equals

$$x_{12}(\alpha\theta_0)x_{13}\left(\frac{-\alpha\theta_0(1+\varepsilon(\eta-1))}{\varepsilon\eta}\right)x_{14}\left(\frac{\theta_0(\alpha(\varepsilon-1)+\varepsilon)}{\varepsilon}\right)x_{23}\left(\frac{-\theta_0}{\varepsilon\eta}\right)x_{24}\left(\frac{-\theta_0}{\varepsilon}\right)x_{34}(-\theta_0),$$

where $\theta_0=(\varepsilon-1)(\eta-1)$.

Suppose that $\operatorname{Char} K\neq 2$. Calculate the following product:

$$\left[z\left(\frac{1}{\eta},1-\eta\right),z\left(\eta,\frac{\eta-1}{\eta}\right)\right]=x_{13}\left(\frac{\alpha\left(\eta^2-1\right)}{\eta}\right)x_{24}\left(\frac{\left(\eta^2-1\right)}{\eta}\right),$$

we get that the subgroup $X_{13,24}^{\alpha}$ is contained in H. Then for $\varepsilon+\eta\neq 3$, $\varepsilon\neq\eta$, we have

$$\left[z\left(\frac{\alpha}{\alpha+1},\varepsilon\right),z\left(\frac{\alpha}{\alpha+1},\eta\right)\right]$$
$$=x_{13}\left(\frac{\theta_0(\eta-\varepsilon)}{(\alpha+1)\varepsilon\eta}\right)x_{24}\left(\frac{\theta_0(\eta-\varepsilon)}{\alpha(\alpha+1)\varepsilon\eta}\right)x_{14}\left(\frac{\theta_0\left(\varepsilon^2-3\varepsilon-\eta+3\eta^2\right)}{(\alpha+1)^2\varepsilon\eta}\right).$$

Multiplying the product $\left[z\left(\frac{\alpha}{\alpha+1}, \varepsilon\right), z\left(\frac{\alpha}{\alpha+1}, \eta\right)\right]$ by suitable elememt of $X^{\alpha}_{13,24}$ we get the whole group X_{14}. For $\varepsilon \neq 2, -2$, $\varepsilon \neq 3\alpha/(3\alpha+2)$, we have

$$z\left(\varepsilon, \frac{\varepsilon-1}{\varepsilon}\right) z\left(\varepsilon, \frac{\varepsilon-1}{\varepsilon-2}\right) = x_{13}\left(-\frac{\alpha(\varepsilon-1)(\varepsilon+2)}{(\varepsilon-2)\varepsilon^2}\right) x_{24}\left(-\frac{(\varepsilon-1)(\varepsilon+2)}{(\varepsilon-2)\varepsilon^2}\right)$$
$$\times\, x_{14}\left(\frac{(\varepsilon-1)(3\alpha(\varepsilon-1)+2\varepsilon)}{(\varepsilon-2)\varepsilon^2}\right) x_{12}\left(\frac{2\alpha(\varepsilon-1)}{(\varepsilon-2)\varepsilon}\right) x_{34}\left(-\frac{2(\varepsilon-1)}{(\varepsilon-2)\varepsilon}\right).$$

Multiplying the product $z\left(\varepsilon, \frac{\varepsilon-1}{\varepsilon}\right)z\left(\varepsilon, \frac{\varepsilon-1}{\varepsilon-2}\right)$ by suitable elements of $X^{\alpha}_{13,24}$ and X_{14}, we can extract an element of the subgroup $X^{-\alpha}_{12,34}$. Next, multiplying $z(\varepsilon, \eta)$ by suitable elements of $X^{-\alpha}_{12,34}$, X_{14} and $X^{\alpha}_{13,24}$, we extract an element $x_{23}\left((\varepsilon-1)(\eta-1)/\varepsilon\eta\right)$. From the decomposition of $y(\eta)$, we get the subgroups Q_{13} and X_{23}.

Note that the characteristic of the field plays role only in extracting the subgroup $X^{-\alpha}_{12,34}$, for this reason for Char $K = 2$ it is sufficient to show that $X^{-\alpha}_{12,34}$ is contained in H. In fact, multiplying the product $z\left(\varepsilon, \frac{\varepsilon-1}{\varepsilon}\right) z\left(\varepsilon+1, \frac{\varepsilon^2}{\varepsilon^2-\varepsilon-1}\right)$ by suitable elements of $X^{\alpha}_{13,24}$ and X_{14}, we can extract an element of the subgroup $X^{-\alpha}_{12,34}$. Therefore we conclude that $H = Q_{13}X^{-\alpha}_{12,34}X_{14}X_{23}X^{\alpha}_{13,24}$, if $\alpha+\beta = 0$.

(5) For the base (r3s4a), $\alpha \neq 0, -1$, $\lambda = 0$, $\beta = -1$. The generators are as follows:

$$x(\varepsilon) = d_1(\varepsilon)d_2(\varepsilon)x_{13}\left(\frac{\alpha(\varepsilon-1)}{\varepsilon}\right) x_{14}\left(\frac{\varepsilon-1}{\varepsilon}\right) x_{23}\left(\frac{1-\varepsilon}{\varepsilon}\right), \quad y(\eta) = d_1(\eta)d_2(\eta)x_{31}(\eta-1).$$

Now suppose that Char $K \neq 2$, $\alpha \neq -\frac{1}{2}$. Then $z_x\left(\varepsilon, \frac{\alpha+1}{\alpha}, \eta\right)$ equals

$$x_{13}\left((2\alpha+1)\varepsilon\theta_0\right) x_{14}\left(\frac{(1+2\alpha)\varepsilon\theta_0}{\alpha}\right) x_{23}\left(-\frac{(2\alpha+1)\varepsilon\theta_0}{\alpha}\right) x_{24}\left(\frac{\theta_0(1-2\alpha\varepsilon-\varepsilon)}{\alpha^2}\right),$$

$$z_x\left(\frac{1}{2\alpha+1}, \frac{\alpha+1}{\alpha}, \eta\right)^2 = x_{13}\left(\frac{4\alpha(1-\eta)}{2\alpha+1}\right) x_{14}\left(\frac{4(1-\eta)}{2\alpha+1}\right) x_{23}\left(\frac{4(\eta-1)}{2\alpha+1}\right).$$

It follows that $X^{-\alpha,-1}_{13,14,23}$ is contained in H. Multiplying $z_x\left(\varepsilon, \frac{1+\alpha}{\alpha}, \eta\right)$ by suitable element from $X^{-\alpha,-1}_{13,14,23}$, we can get an element $x_{24}\left(-\theta_0(2\alpha\varepsilon+\varepsilon-1)/\alpha^2\right)$. Take $\varepsilon \neq -1/(2\alpha+1)$, we obtain all elements of X_{24}.

Suppose that $\alpha = -\frac{1}{2}$, direct calculation shows that

$$z_x\left(\frac{1}{2}, 3, \eta\right)^2 = x_{13}(1-\eta)x_{14}(2-2\eta)x_{23}(-2+2\eta),$$

therefore $X^{-\frac{1}{2},-1}_{13,14,23} \leqslant H$. Calculate the product:

$$z_x\left(\varepsilon, 3, \eta\right) x_{13}(-2\varepsilon\theta_0)x_{24}(-4\varepsilon\theta_0)x_{23}(4\varepsilon\theta_0) = x_{24}(-4\left(2\varepsilon^2-3\varepsilon+1\right)(\eta-1)),$$

where $\theta_0 = (\varepsilon-1)(\eta-1)$. Take $\varepsilon \neq \frac{1}{2}$, we get all elements of X_{24}.

In the case of characteristic 2,

$$z_x\left(\varepsilon, \frac{\alpha+1}{\alpha}, \eta\right) z_x\left(\varepsilon+1, \frac{\alpha+1}{\alpha}, \frac{\varepsilon^2\eta+\eta+1}{\varepsilon^2}\right) = x_{13}\left(\frac{\theta_2}{\varepsilon}\right) x_{14}\left(\frac{\theta_2}{\alpha\varepsilon}\right) x_{23}\left(\frac{\theta_2}{\alpha\varepsilon}\right),$$

where $\theta_2 = (\varepsilon+1)(\eta+1)$. It follows that $X^{\alpha}_{13,14,23}$ is subgroup of H. Multiplying $z_x\left(\varepsilon, \frac{\alpha+1}{\alpha}, \eta\right)$ by suitable element from $X^{\alpha}_{13,14,23}$, we get an element of the subgroup X_{24}. From the decomposition of the generators, we consecutively get Q_{13} and X_{31}.

Now describe the subgroup $\langle X^{\alpha}_{13,14,23}, X_{31}\rangle$. Put $t(\varepsilon) = x_{13}(\varepsilon)x_{14}(\varepsilon/\alpha)\,x_{23}(\varepsilon/\alpha)$, $s(\eta) = x_{31}(\eta)$. Note that the elements $t(\varepsilon)$ and $s(\eta)$ lie in H. Consider the map ϕ from the group generated by $t(\varepsilon)$ and $s(\eta)$ to $\mathrm{SL}(2,K)$ defined by $\phi(t(\varepsilon)) = x_{12}(\varepsilon)$, $\phi(s(\eta)) = x_{21}(\eta)$. It is clear that ϕ is a surjective homomorphism. Thus $\langle t(\varepsilon), s(\eta), \varepsilon, \eta \in K\rangle$ is isomorphic to $\operatorname{Ker}\phi \rtimes \mathrm{SL}(2,K)$.

We set $w(\xi) = t(\xi)s(-\xi^{-1})t(\xi)$, $h(\xi) = w_(\xi)w(1)^{-1}$. It is clear that

$$\operatorname{Ker}\phi = \left\langle w(\xi)t(\varepsilon)w(-\xi)s(\xi^{-2}\varepsilon),\ h(\xi)h(\zeta)h(\xi^{-1}\zeta^{-1}),\ \xi,\ \zeta \in K^*\right\rangle.$$

Direct calculation yields that $\operatorname{Ker}\phi = X_{24}$. Then we get $\langle X^{-\alpha,-1}_{13,14,23}, X_{31}\rangle$ is isomorphic to $\mathrm{SL}(2,K)X_{24}$. Therefore we conclude that $H = \mathrm{GL}(2,K)X_{24}$.

• For the base (r3s4a), $\alpha = 0$, $\lambda = 1$, $\beta \neq 0, -1$. Conjugating by $x_{12}(1)$ results in the generators given below:

$$x(\varepsilon) = d_1(\varepsilon)d_2(\varepsilon)x_{13}\left(\frac{\beta(\varepsilon-1)}{\varepsilon}\right)x_{14}\left(\frac{\varepsilon-1}{\varepsilon}\right)x_{23}\left(\frac{\beta(\varepsilon-1)}{\varepsilon}\right), \quad y(\eta) = d_1(\eta)d_2(\eta)x_{31}(\eta-1).$$

A minor variation of the argument of the base (r3s4a), $\alpha \neq 0, -1$, $\lambda = 0$, $\beta = -1$ establishes that H is generated by the subgroups Q_{12}, X_{24}, X_{31} and $X^{\beta,\beta}_{13,23,14}$. Further we consider the subgroup generated by a pair of subgroups X_{31} and $X^{\beta,\beta}_{13,23,14}$, a similar argument for $\langle X_{31}, X^{\beta,\beta}_{13,23,14}\rangle$ yields that $H = \mathrm{GL}(2,K)X_{24}$.

• For the base (r3s4a), $\alpha \neq 0, -1$, $\lambda = 1$, $\beta = -1$. Let $\alpha \neq 1$. Suppose that $\operatorname{Char} K \neq 2$, a simultaneous conjugation by the element $x_{12}(1)$ leads to the following generators:

$$x(\varepsilon) = d_1(\varepsilon)d_2(\varepsilon)x_{13}\left(\frac{(\alpha-1)(\varepsilon-1)}{\varepsilon}\right)x_{14}\left(\frac{\varepsilon-1}{\varepsilon}\right)x_{23}\left(\frac{1-\varepsilon}{\varepsilon}\right),$$
$$y(\eta) = d_1(\eta)d_2(\eta)x_{31}(\eta-1).$$

For $\alpha \neq \frac{1}{2}$, $\eta \in K^*$, straightforward calculations show that $z_x\big(\varepsilon, \frac{\alpha}{\alpha-1}, \eta\big)$ equals

$$x_{13}\left((2\alpha-1)\varepsilon\theta_0\right)x_{14}\left(\frac{(2\alpha-1)\varepsilon\theta_0}{\alpha-1}\right)x_{23}\left(-\frac{(2\alpha-1)\varepsilon\theta_0}{\alpha-1}\right)x_{24}\left(-\frac{\theta_0((2\alpha-1)\varepsilon-1)}{(\alpha-1)^2}\right),$$
$$z_y\left(\frac{\alpha-1}{\alpha}, \varepsilon, \eta\right) = x_{24}\left(-\frac{\theta_0}{(\alpha-1)\alpha}\right)x_{31}\left(\frac{(2\alpha-1)\theta_0}{(\alpha-1)\eta}\right),$$
$$z_x\left(\frac{1}{2\alpha-1}, \frac{\alpha}{\alpha-1}, \eta\right)^2 = x_{13}\left(-\frac{4(\alpha-1)(\eta-1)}{2\alpha-1}\right)x_{14}\left(-\frac{4(\eta-1)}{2\alpha-1}\right)x_{23}\left(\frac{4(\eta-1)}{2\alpha-1}\right),$$

where $\theta_0 = (\varepsilon-1)(\eta-1)$. It follows that $X^{1-\alpha,-1}_{13,14,23} \leqslant H$, then from the decomposition of the generators and $z_y\left((\alpha-1)/\alpha, \varepsilon, \eta\right)$ we obtain Q_{12}, X_{31} and X_{24}. Straightforward calculations show that $H = \langle Q_{12}, X^{1-\alpha,-1}_{13,14,23}, X_{31}, X_{24}\rangle = \mathrm{GL}(2,K)X_{24}$.

Suppose that $\alpha = \frac{1}{2}$, simultaneous conjugation by $x_{12}(1)$ yields the following generators:

$$x(\varepsilon) = d_1(\varepsilon)d_2(\varepsilon)x_{13}\left(\frac{1-\varepsilon}{2\varepsilon}\right)x_{14}\left(\frac{\varepsilon-1}{\varepsilon}\right)x_{23}\left(\frac{1-\varepsilon}{\varepsilon}\right), \quad y(\eta) = d_1(\eta)d_2(\eta)x_{31}(\eta-1).$$

The calculation straightforwardly gives that $z_y(-1, \varepsilon, \eta) = x_{24}(4(\varepsilon-1)(\eta-1))$. Put

$$f(\varepsilon,\eta) = x(\varepsilon)y\left(\frac{-\varepsilon\eta-\varepsilon+\eta+1}{(\varepsilon+1)(\eta-1)}\right)x(\eta), \quad g(\varepsilon,\eta) = y(\varepsilon)x\left(\frac{-\varepsilon\eta+\varepsilon-\eta+1}{(\varepsilon-1)(\eta+1)}\right)y(\eta).$$

Multiplying $[f(\varepsilon,\eta), f(\eta,\varepsilon)]$ and $[g(\varepsilon,\eta), g(\eta,\varepsilon)]$ by suitable element from X_{24}, we get the elements from X_{31} and $X_{13,14,23}^{\frac{1}{2},-1}$. Thus we can conclude that $H = \mathrm{GL}(2,K)X_{24}$.

In the case of characteristic 2, we have

$$r(\varepsilon,\eta) = z_x\left(\varepsilon, \frac{\alpha}{\alpha+1}, \eta\right) = x_{13}\left(\varepsilon\theta_2\right) x_{14}\left(\frac{\varepsilon\theta_2}{\alpha+1}\right) x_{23}\left(\frac{\varepsilon\theta_2}{\alpha+1}\right) x_{24}\left(\frac{(\varepsilon+1)\theta_2}{(\alpha+1)^2}\right),$$

$$r(\eta,\theta)r\left(\varepsilon, \frac{\varepsilon^2+\eta^2\theta+\eta^2+\theta}{\varepsilon^2+1}\right) = x_{13}\left(\frac{\theta_7}{(\varepsilon+1)}\right) x_{14}\left(\frac{\theta_7}{(\alpha+1)(\varepsilon+1)}\right) x_{23}\left(\frac{\theta_7}{(\alpha+1)(\varepsilon+1)}\right),$$

where $\theta_2 = (\varepsilon+1)(\eta+1)$, $\theta_7 = \varepsilon(\eta(\theta+1)+\theta+1)+\eta(\eta+1)(\theta+1)$. We get the subgroup $X_{13,14,23}^{1+\alpha}$, after it we get that X_{31} and Q_{12} are contained in H. In all case we can conclude that $H = \mathrm{GL}(2,K)X_{24}$.

- For the base (r3s4a), $\alpha \neq 0,-1$, $\lambda = 1$, $\beta \neq 0,-1$, $1+\alpha+\beta \neq 0$. If $\alpha+\beta \neq 0$, conjugating this base by the element $x_{23}\big(\frac{-\beta}{\alpha+\beta}\big)x_{32}\big(\frac{\alpha+\beta}{\beta}\big)d_1(\beta)w_{13}x_{12}\big(-\frac{\alpha}{\beta}\big)$, then generators have the following form:

$$x(\varepsilon) = d_2(\varepsilon)d_3(\varepsilon)x_{24}\left(\frac{-\beta(\varepsilon-1)}{\alpha+\beta}\right) x_{31}\left(\frac{(\alpha+\beta)(\varepsilon-1)}{\beta\varepsilon}\right) x_{34}\left(\frac{\varepsilon-1}{\varepsilon}\right),$$
$$y(\eta) = d_2(\eta)d_3(\eta)x_{13}(\beta(\eta-1)).$$

Suppose that $\mathrm{Char}\,K \neq 2$, when $\alpha+\beta \neq -\frac{1}{2}$, we have

$$\left[z\left(\varepsilon, \frac{\alpha+\beta+1}{\alpha+\beta}\right), z\left(\eta, \frac{\alpha+\beta+1}{\alpha+\beta}\right)\right] = x_{31}\left(\frac{\theta_8}{\beta}\right) x_{34}\left(\frac{\theta_8}{\alpha+\beta}\right),$$

$$\left[z\left(\frac{\alpha+\beta}{\alpha+\beta+1}, \varepsilon\right), z\left(\frac{\alpha+\beta}{\alpha+\beta+1}, \eta\right)\right] = x_{13}\left(\frac{\beta\theta_8}{\alpha+\beta}\right),$$

where $\theta_8 = (\varepsilon-1)(\eta-1)(2\alpha+2\beta+1)(\varepsilon-\eta)$. When $\alpha+\beta = -\frac{1}{2}$, we put

$$f(\varepsilon,\eta) = z_x\left(\varepsilon, \eta, \frac{-\varepsilon^2\eta^2+\varepsilon^2-2\varepsilon\eta-2\varepsilon+\eta^2-2\eta+1}{\varepsilon\left(\varepsilon^2\eta^2-2\varepsilon^2\eta+\varepsilon^2-2\varepsilon\eta-2\varepsilon-\eta^2+1\right)}\right),$$

$$g(\varepsilon,\eta) = z_y\left(\varepsilon, \eta, \frac{\varepsilon\eta+\varepsilon-\eta+1}{\varepsilon\eta+\varepsilon+\eta-1}\right).$$

Calculations $[f(\varepsilon_1,\eta_1), f(\varepsilon_2,\eta_2)]$ and $[g(\varepsilon_1,\eta_1), g(\varepsilon_2,\eta_2)]$ result in the subgroups X_{13} and $X_{31,34}^{-\frac{1}{2\beta}}$.

In the case of characteristic 2, for $\varepsilon \neq \eta$, $\theta = (\varepsilon+1)(\eta+1)$,

$$\left[z\left(\varepsilon, \frac{\alpha+\beta+1}{\alpha+\beta}\right), z\left(\eta, \frac{\alpha+\beta+1}{\alpha+\beta}\right)\right] = x_{31}\left(\frac{\theta(\varepsilon+\eta)}{\beta}\right) x_{34}\left(\frac{\theta(\varepsilon+\eta)}{\alpha+\beta}\right),$$

$$\left[z\left(\frac{\alpha+\beta}{\alpha+\beta+1}, \varepsilon\right), z\left(\frac{\alpha+\beta}{\alpha+\beta+1}, \eta\right)\right] = x_{13}\left(\frac{\beta\theta(\varepsilon+\eta)}{\alpha+\beta}\right).$$

Therefore, in all cases it follows from the decomposition of the generators that the subgroups X_{13}, Q_{23}, $X_{31,34}^{\frac{\alpha+\beta}{\beta}}$ and X_{24} are contained in H. As previous arguments, the subgroup $\left\langle X_{13}, X_{31,34}^{\frac{\alpha+\beta}{\beta}}\right\rangle$ is isomorphic to $\mathrm{SL}(2,K)$, and then we obtain that $H = \mathrm{GL}(2,K)X_{24}$, if $\alpha+\beta \neq 0$.

(6) For the base (r3s4a), $\alpha = -1$, $\lambda = 1$, $\beta = -1$. If $\operatorname{Char} K \neq 2, 3$, new generators become the following form:

$$x(\varepsilon) = d_1(\varepsilon)d_3(\varepsilon)x_{12}\left(\frac{1-\varepsilon}{\varepsilon}\right)x_{34}\left(\frac{\varepsilon-1}{\varepsilon}\right)x_{32}\left(\frac{2(1-\varepsilon)}{\varepsilon}\right),$$

$$y(\eta) = d_1(\eta)d_3(\eta)x_{23}(\eta - 1).$$

Due to straightforward calculations, we find that

$$\left[z_x\left(-\frac{1}{3}, \frac{1}{2}, \varepsilon\right), x(\eta)\right] = x_{12}\left(\frac{2\theta_0}{3}\right)x_{34}\left(-\frac{2\theta_0}{3}\right)x_{32}\left(\frac{4\theta_0}{3}\right),$$

$$z_y\left(2, \varepsilon, \eta\right) = x_{14}\left(\frac{-\theta_0}{2}\right)x_{23}\left(\frac{3\theta_0}{2\eta}\right),$$

where $\theta_0 = (\varepsilon - 1)(\eta - 1)$. It follows that $X_{12,34,32}^{\frac{1}{2},-\frac{1}{2}}$ is contained in H. We consecutively get Q_{13} and X_{23} from the decomposition of the generators. Further, we can extract an element of X_{14} through multiplying $z_y(2, \varepsilon, \eta)$ by suitable element from X_{23}. Hence for some $\theta \in K^*$, we may get the whole group X_{14}.

Now describe the subgroup $\left\langle X_{12,34,32}^{\frac{1}{2},-\frac{1}{2}}, X_{23}\right\rangle$. Consider a map ϕ from the subgroup generated by $t(\varepsilon)$ and $s(\eta)$ to $\mathrm{SL}(2, K)$ defined by $\phi(t(\varepsilon)) = x_{23}(\varepsilon)$, $\phi(s(\eta)) = x_{32}(\eta)$. Note that $t(\varepsilon) = x_{32}(\varepsilon)x_{12}\left(\frac{\varepsilon}{2}\right)x_{34}\left(-\frac{\varepsilon}{2}\right)$, $s(\eta) = x_{23}(\eta)$. A straightforward calculation shows that $\operatorname{Ker}\phi = X_{14}$. Therefore we conclude that $H = \mathrm{GL}(2, K)X_{14}$, if $\operatorname{Char} K \neq 2, 3$.

If $\operatorname{Char} K = 3$, the original generators become the following form:

$$x'(\varepsilon) = d_1(\varepsilon)d_2(\varepsilon)x_{13}\left(\frac{2(\varepsilon-1)}{\varepsilon}\right)x_{14}\left(\frac{\varepsilon-1}{\varepsilon}\right)x_{23}\left(\frac{2(\varepsilon-1)}{\varepsilon}\right),$$

$$y'(\eta) = d_1(\eta)d_2(\eta)x_{31}(\eta - 1)x_{32}(\eta - 1).$$

Conjugating this base by the element $w_{23}w_{12}x_{12}(1)$ we get new generators

$$x(\varepsilon) = d_1(\varepsilon)d_3(\varepsilon)x_{12}\left(\frac{2(\varepsilon-1)}{\varepsilon}\right)x_{34}\left(\frac{\varepsilon-1}{\varepsilon}\right)x_{32}\left(\frac{\varepsilon-1}{\varepsilon}\right),$$

$$y(\eta) = d_1(\eta)d_3(\eta)x_{23}(\eta - 1).$$

Due to straightforward calculations, we find that $z_x\left(\varepsilon, 2, \eta\right) = x_{14}(2(2\varepsilon+1)(2+\eta))$. It follows that X_{14} is contained in H. Put

$$Q(\varepsilon, \eta) = x\left(\frac{\varepsilon\eta + 2\varepsilon + 2\eta + 2}{2\varepsilon\eta + 2\varepsilon + \eta + 2}\right)y(\eta)x(\varepsilon)x_{14}\left(\frac{2(\varepsilon+2)^2(\eta+2)}{\varepsilon(\varepsilon\eta + 2\varepsilon + 2\eta + 2)}\right).$$

Calculate the commutator

$$\left[Q(\varepsilon, 1), Q\left(\frac{2\eta+1}{2(\eta+1)}, \eta\right)\right] = x_{23}\left(\frac{2\varepsilon}{2\eta+1}\right).$$

Thus we obtain the subgroup X_{23}. It follows that from the decomposition of the generators we extract the subgroups $X_{12,34,32}^2$ and Q_{13}.

Now describe the subgroup $\left\langle X_{12,34,32}^2, X_{23}\right\rangle$. As above argument, we may get $\left\langle X_{12,34,32}^2, X_{23}\right\rangle$ is isomorphic to $\mathrm{SL}(2, K)X_{14}$, and then conclude that $H = \mathrm{GL}(2, K)X_{14}$, when $\operatorname{Char} K = 3$.

(7) For the base (r3s4a), $\alpha=-1$, $\lambda=1$, $\beta\neq 0,-1$. Let $\beta\neq 1$, suppose that $\operatorname{Char}K\neq 2$. New generators are obtained with the help of conjugation by $d_1(\beta)w_{13}x_{12}\left(\frac{1}{\beta}\right)$. Then

$$x(\varepsilon)=d_2(\varepsilon)d_3(\varepsilon)x_{21}\left(\frac{\varepsilon-1}{\varepsilon}\right)x_{34}\left(\frac{\varepsilon-1}{\varepsilon}\right),$$

$$y(\eta)=d_2(\eta)d_3(\eta)x_{12}\left(\frac{(\beta-1)(\eta-1)}{\varepsilon}\right)x_{13}(\beta(\eta-1)).$$

When $\beta\neq\frac{1}{2}$, for the element $\eta\in K^*$, by straightforward calculation, we see that

$$z_x\left(\frac{1}{2\beta-1},\frac{\beta}{\beta-1},\eta\right)^2=x_{21}\left(-\frac{4(\eta-1)}{2\beta-1}\right)x_{34}\left(-\frac{4(\eta-1)}{2\beta-1}\right),$$

$$z_y\left(\frac{\beta-1}{\beta},\varepsilon,\eta\right)=x_{12}\left(\frac{(2\beta-1)\theta_0}{\eta}\right)x_{13}\left(\frac{\beta(2\beta-1)\theta_0}{\eta(\beta-1)}\right)x_{24}\left(\frac{\theta_0}{\beta-1}\right)x_{34}\left(\frac{-\theta_0}{\beta}\right),$$

where $\theta_0=(\varepsilon-1)(\eta-1)$. It follows that $X_{21,34}\leqslant H$. Note that we also get $X_{21,34}$, when $\operatorname{Char}K=3$. In fact, when $\operatorname{Char}K=3$, we have

$$z_x\left(\frac{1}{2\beta+2},\frac{\beta}{\beta-1},\theta\right)^2=x_{21}\left(\frac{\theta-1}{\beta+1}\right)x_{34}\left(\frac{\theta-1}{\beta+1}\right).$$

Then from the decomposition of the generators we obtain Q_{23} and $X_{12,13}^{\frac{\beta-1}{\beta}}$. After it we have the subgroup $X_{24,34}^{\frac{\beta}{1-\beta}}$. Straightforward calculations show that $\left\langle X_{21,34},X_{12,13}^{\frac{\beta-1}{\beta}}\right\rangle\cong\mathrm{SL}(2,K)X_{24,34}^{\frac{\beta}{1-\beta}}$, and finally we conclude that $H=\mathrm{GL}(2,K)X_{24,34}^{\frac{\beta}{1-\beta}}$.

Suppose that $\beta=\frac{1}{2}$, a simultaneous conjugation by the element $w_{12}d_1\left(\frac{1}{2}\right)w_{13}x_{12}(2)$ leads to the following generators:

$$x(\varepsilon)=d_1(\varepsilon)d_3(\varepsilon)x_{12}\left(\frac{\varepsilon-1}{\varepsilon}\right)x_{34}\left(\frac{1-\varepsilon}{\varepsilon}\right),\quad y(\eta)=d_1(\eta)d_3(\eta)x_{21}\left(\frac{1-\eta}{2}\right)x_{23}\left(\frac{\eta-1}{2}\right).$$

We have $z_y\left(-1,\eta,\theta\right)=x_{14}(-2(\eta-1)(\theta-1))x_{34}(-2(\eta-1)(\theta-1))$. Put

$$f(\varepsilon,\eta)=x(\varepsilon)y\left(\frac{(1-\varepsilon)(1+\eta)}{(\varepsilon+1)(\eta-1)}\right)x(\eta),\quad g(\varepsilon,\eta)=y(\varepsilon)x\left(\frac{(1+\varepsilon)(1-\eta)}{(\varepsilon-1)(\eta+1)}\right)y(\eta).$$

Multiplying $[f(\varepsilon,\eta),f(\eta,\varepsilon)]$ and $[g(\varepsilon,\eta),g(\eta,\varepsilon)]$ by suitable element from $X_{14,34}$, we get the elements, $x_{21}(\theta_9)x_{23}(-\theta_9)$, $x_{12}(-\theta_{10})x_{14}(\theta_{10})$, where

$$\theta_9=\frac{(\varepsilon\eta-1)^2\left(\varepsilon^2-\eta^2\right)}{(\varepsilon-1)\varepsilon(\varepsilon+1)(\eta-1)\eta(\eta+1)},\quad \theta_{10}=\frac{2(\varepsilon\eta-1)^2\left(\varepsilon^2-\eta^2\right)}{(\varepsilon-1)\varepsilon(\varepsilon+1)(\eta-1)\eta(\eta+1)}.$$

Take $\varepsilon,\eta\neq-1$, $\varepsilon\eta\neq 1$ and $\varepsilon^2\neq\eta^2$, we may get all elements of $X_{21,23}^{-1}$ and $X_{12,14}^{-1}$. From the decomposition of the generators, we get that Q_{13} and $X_{12,34}^{-1}$ are subgroups of H. And $\left\langle Q_{13},X_{12,34}^{-1},X_{21,23}^{-1}\right\rangle\cong\mathrm{GL}(2,K)X_{14,34}$. Thus we can conclude that $H=\mathrm{GL}(2,K)X_{14,34}$.

In the case of characteristic 2, we use the following generators:

$$x(\varepsilon)=d_2(\varepsilon)d_3(\varepsilon)x_{21}\left(\frac{\varepsilon-1}{\varepsilon}\right)x_{34}\left(\frac{\varepsilon-1}{\varepsilon}\right),$$

$$y(\eta)=d_2(\eta)d_3(\eta)x_{12}\left(\frac{(\beta-1)(\eta-1)}{\varepsilon}\right)x_{13}(\beta(\eta-1)).$$

Then we have

$$r(\varepsilon,\eta)=z_y\left(\frac{\beta+1}{\beta},\varepsilon,\eta\right)=x_{12}\left(\frac{\theta_2}{\eta}\right)x_{13}\left(\frac{\beta\theta_2}{(\beta-1)\eta}\right)x_{24}\left(\frac{\theta_2}{\beta+1}\right)x_{34}\left(\frac{\theta_2}{\beta}\right),$$

$$r(\eta,\theta)r\left(\varepsilon,\frac{\varepsilon^2+\varepsilon+\eta^2\theta+\eta^2+\eta\theta+\eta}{(\varepsilon+1)\varepsilon}\right)=x_{12}\left(\frac{\theta_2(\theta+1)}{\varepsilon\theta}\right)x_{13}\left(\frac{\beta\theta_2(\theta+1)}{(\beta-1)\varepsilon\theta}\right),$$

where $\theta_2=(\varepsilon+1)(\eta+1)$. We get the subgroup $X_{13,12}^{\frac{\beta}{\beta-1}}$, after it we obtain $X_{21,34}$ and Q_{23} are contained in H, and $\left\langle Q_{23},X_{21,34},X_{13,12}^{\frac{\beta}{\beta-1}}\right\rangle\cong \mathrm{GL}(2,K)X_{24,34}^{\frac{\beta}{1-\beta}}$. Thus we can conclude that $H=\mathrm{GL}(2,K)X_{24,34}^{\frac{\beta}{1-\beta}}$.

(8) For the base (p1q1a). Put

$$g_1=w_{12},\quad g_2=x_{12}(-1)x_{24}(1)w_{12},\quad g_3=w_{12}x_{13}(1),\quad g_4=x_{14}(-1)x_{23}(1)x_{12}(-1)w_{12}.$$

We have four cases depending on the value of λ_i, $i=1,2$. (I) $\lambda_1=0$, $\lambda_2=0$, (II) $\lambda_1=0$, $\lambda_2=1$, (III) $\lambda_1=1$, $\lambda_2=0$ and (VI) $\lambda_1=1$, $\lambda_2=1$. We conjugate the corresponding generators by g_1, g_2, g_3 and g_4 respectively, and get the new generators, which are upper triangular matrices. Then we calculate commutator subgroup $z(\varepsilon,\eta)$ generated by new generators $x(\varepsilon)$ and $y(\eta)$. Moreover, it follows from the decomposition of $x(\varepsilon)$ and $y(\eta)$ that in all cases the subgroups Q_{12}, Q_{23} and X_{24} are contained in H. Thus we conclude that $H=Q_{12}Q_{23}X_{24}$. ▷

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ПОДГРУППЫ, ПОРОЖДЕННЫЕ ПАРОЙ 2-ТОРОВ В $GL(4, K)$. II

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Аннотация. Данная статья является очередной работой в большом цикле работ, посвященном геометрии микровесовых торов в группах Шевалле. А именно, мы описываем подгруппы, порожденные парой 2-торов в $\mathrm{GL}(4, K)$. Напомним, что 2-торами в $\mathrm{GL}(n, K)$ являются подгруппы, сопряженные диагональной подгруппе вида $\mathrm{diag}(\varepsilon, \varepsilon, 1, \dots, 1)$. В одной из предыдущих работ мы доказали теорему редукции для пары m-торов. Из нее следует, что любая пара 2-торов может быть вложена в $\mathrm{GL}(6, K)$ одновременным сопряжением. Орбита пары 2-торов (X, Y) называется орбитой в $\mathrm{GL}(n, K)$, если пара (X, Y) вкладывается в $\mathrm{GL}(n, K)$ одновременным сопряжением и не вкладывается в $\mathrm{GL}(n-1, K)$. Здесь n может принимать значения 3, 4, 5 и 6. Наиболее сложным и общим случаем является случай $\mathrm{GL}(4, K)$. В настоящей работе описаны порождения в $\mathrm{GL}(4, K)$, соответствующие вырожденным орбитам.

Ключевые слова:полная линейная группа, унипотентная корневая подгруппа, полупростые корневые подгруппы, m-торы, диагональные подгруппы.

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