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ON SOME INTERPOLATION INEQUALITIES DUE TO OLGA LADYZHENSKAYA AND NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. We consider some multiplicative interpolation inequalities between the Hölder space and the Lebesgue space. Multiplicative interpolation inequalities of the Gagliardo–Nirenberg type are used in the investigations of partial differential equations. Several such inequalities involving the Hölder norm (seminorm) were already proved and applied. In the present paper we generalise previous results to the anisotropic "parabolic" case with another simple proof due to idea of Olga Ladyzhenskaya. The manuscript also contains an application of such Gagliardo–Nirenberg type inequality with the Hölder norm. Some integral estimate and this inequality give a priori estimate of the solution to quasilinear parabolic problem in the smooth Hölder classes. Moreover, using this a priori estimate, we establish the existence of solution of the quasilinear parabolic problem. In order to prove multiplicative inequality of the Gagliardo–Nirenberg type with the Hölder norm we use an equivalent normalization of the higher order Hölder spaces over higher order finite differences. The key technical tool is the representation of a function $u(x,t)$ at an arbitrary fixed point (x,t) over a higher order finite difference at this point and the corresponding additional sum of values at neighboring points. After that we integrate with respect to the neighboring points over the balls $B_r((x,t))$ of small radius r. Estimating the finite difference over the corresponding Hölder seminorm, we obtain an additive inequality with the parameter r, involving the Hölder and integral norms. Optimizing this inequality over r we get the multiplicative estimate of the Gagliardo–Nirenberg type with the Hölder norm and the Lebesgue norm.

Keywords: interpolation inequalities, a-priori estimates, nonlinear PDE.

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1. Introduction

In the present paper we deal with some simple interpolation inequalities between the Hölder and the Lebesgue spaces. The importance of interpolation inequalities in the investigations of different problems for partial differential equations is widely known, see, for example, [1, Ch. 5, Ch. 6]. The inequalities from this paper can be used, for example, in some appropriate situations to obtain a priori estimate in the Hölder spaces when the estimate in a particular the Lebesgue space has already been obtained. The last estimate can very often be obtained more easily. We give a simple example of an application of the interpolation inequalities from this paper to a priori estimate for a solutions to some quasilinear parabolic initial boundary value problem.

The subject of interpolation inequalities is so vast that it is impossible to describe even short history of this question, so we confine ourselves to refereing to books [1, Ch. 5, Ch. 6; 2, Ch. 1; 3, from section 1.6 and throughout], and the references therein.

From the previous papers with aggregate results on the interpolation of norms between different scales of function spaces we mention only the papers [4–10], where the more comprehensive bibliography can be found. Here the papers [4] and [5] deal with interpolation inequalities in the Lebesgue and the Sobolev spaces involving the Hölder norms (similar to the present paper). Moreover, the papers [6, 7] contain multiplicative estimates of the Lebesgue and the Sobolev norms over the Lebesgue and the Besov norms, including the limiting the Hölder case $B^{\gamma}_{\infty,\infty} = C^{\gamma}$. On the other hand, in the papers [8, 9] multiplicative interpolation inequalities between the Hölder (or BMO) and the Lorentz norms of a function are proved. Besides, inequalities from [9] bind together the Lebesgue or the Sobolev, the Lorentz and the Besov–Lorentz norms. Mention also that the inequalities from the paper [10] include BMO norm.

Thus the present paper can be viewed as an extension of the results in [4–10] to the anisotropic “parabolic” case and with another proof. Moreover, the present paper is motivated, in particular, by [11], where the simplest situation was considered, and where we got the idea of simple proof for such sharp inequalities. Note also that in this paper we consider for simplicity the known functional spaces, designed for parabolic and elliptic equations of the second order.

Let us now give several definitions and auxiliary facts. We are going to use standard spaces $C^{m+\alpha}(\overline{\Omega})$ and $C^{m+\alpha,\frac{m+\alpha}{2}}(\overline{\Omega}_T)$ of the Hölder continuous functions $u(x)$ and $u(x,t)$, where $m = 0, 1, 2, \dots$, $\alpha \in (0,1)$, Ω is a given domain in R^N (bounded or unbounded) with smooth boundary (of the class $C^{m+\alpha}$), $\Omega_T = \Omega \times (0,T)$, $\overline{\Omega}_T = \overline{\Omega} \times [0,T]$, $T > 0$ is a given constant. The norm in the space $C^{m+\alpha}(\overline{\Omega})$ is defined by

$$\|u\|_{C^{m+\alpha}(\overline{\Omega})} \equiv |u|^{(m+\alpha)}_{\overline{\Omega}} = |u|^{(m)}_{\overline{\Omega}} + \langle u\rangle^{(m+\alpha)}_{\overline{\Omega}}, \tag{1}$$

where

$$|u|^{(m)}_{\overline{\Omega}} \equiv \sum_{|\overline{\beta}|\leqslant m} \max_{\overline{\Omega}} \left|D^{\overline{\beta}}_x u(x)\right|, \quad \langle u\rangle^{(m+\alpha)}_{\overline{\Omega}} \equiv \sum_{|\overline{\beta}|=m} \left\langle D^{\overline{\beta}}_x u(x)\right\rangle^{(\alpha)}_{\overline{\Omega}}, \tag{2}$$

$$\left\langle w(x)\right\rangle^{(\alpha)}_{\overline{\Omega}} \equiv \left\langle w(x)\right\rangle^{(\alpha)}_{x,\overline{\Omega}} \equiv \sup_{x,y\in\overline{\Omega}} \frac{|w(x)-w(y)|}{|x-y|^{\alpha}},$$

$\overline{\beta} = (\beta_1, \beta_2, \dots, \beta_N)$ is a multiindex, $\beta_i = 0, 1, 2, \dots$, $|\overline{\beta}| = \beta_1 + \beta_2 + \ldots + \beta_N$,

$$D^{\overline{\beta}}_x u = \frac{\partial^{\beta_1}\partial^{\beta_2}\ldots\partial^{\beta_N} u(x)}{\partial x_1^{\beta_1}\partial x_2^{\beta_2}\ldots\partial x_N^{\beta_N}}.$$

Analogously, the norm in the space $C^{m+\alpha,\frac{m+\alpha}{2}}(\overline{\Omega}_T)$ is defined by (k, l are positive integers)

$$\|u\|_{C^{m+\alpha,\frac{m+\alpha}{2}}(\overline{\Omega}_T)} \equiv |u|^{(m+\alpha)}_{\overline{\Omega}_T} = |u|^{(m)}_{\overline{\Omega}_T} + \langle u\rangle^{(m+\alpha)}_{\overline{\Omega}_T}, \tag{3}$$

$$|u|^{(m)}_{\overline{\Omega}_T} \equiv \sum_{|\overline{\beta}|+2l\leqslant m} \max_{\overline{\Omega}_T} \left|D^l_t D^{\overline{\beta}}_x u(x,t)\right|, \quad \langle u\rangle^{(m+\alpha)}_{\overline{\Omega}_T} = \langle u\rangle^{(m+\alpha)}_{x,\overline{\Omega}_T} + \langle u\rangle^{(m+\alpha)}_{t,\overline{\Omega}_T}, \tag{4}$$

$$\langle u\rangle^{(m+\alpha)}_{x,\overline{\Omega}_T} \equiv \sum_{0\leqslant m-|\beta|-2l\leqslant 1} \left\langle D^l_t D^{\overline{\beta}}_x u(x)\right\rangle^{(\alpha)}_{x,\overline{\Omega}},$$

$$\langle u\rangle^{(m+\alpha)}_{t,\overline{\Omega}_T} \equiv \sum_{0\leqslant m-|\beta|-2l\leqslant 1} \left\langle D^l_t D^{\overline{\beta}}_x u(x)\right\rangle^{\left(\frac{m-|\beta|-2l+\alpha}{2}\right)}_{t,\overline{\Omega}}. \tag{5}$$

Note that for the space $C^{m+\alpha,\frac{m+\alpha}{2}}(\overline{\Omega}_T)$ we also have the estimate (see, for example, [12])

$$\begin{gathered}\langle u\rangle^{(m+\alpha)}_{t,\overline{\Omega}_T}=\sum_{0\leqslant m-|\beta|-2l\leqslant 1}\left\langle D^l_t D^{\overline{\beta}}_x u(x)\right\rangle^{\left(\frac{m-|\beta|-2l+\alpha}{2}\right)}_{t,\overline{\Omega}}\\ \leqslant C\left(\langle u\rangle^{(m+\alpha)}_{x,\overline{\Omega}_T}+\left\langle D^{[m/2]}_t u(x)\right\rangle^{\left(\frac{m-2[m/2]+\alpha}{2}\right)}_{t,\overline{\Omega}}\right),\end{gathered} \tag{6}$$

where here and everywhere below we denote by C and ν all absolute constants or constants depending on fixed data of the problem. It is known (see [3, 5, 13]) that the seminorm $\langle u\rangle^{(m+\alpha)}_{\overline{\Omega}_T}$ is equivalent to the seminorm

$$\langle u\rangle^{(m+\alpha)}_{\overline{\Omega}_T}\simeq C_{k,l}\left(\sup_{\overline{h}\in R^3;\,x,\,x+\overline{h}\in\overline{\Omega};\,t\in[0,\,T]}\frac{\left|\Delta^k_{x,\overline{h}}u(x,t)\right|}{\left|\overline{h}\right|^{m+\alpha}}+\sup_{\Delta t>0;\,x\in\overline{\Omega};\,t\in[0,T]}\frac{\left|\Delta^l_{t,\Delta t}u(x,t)\right|}{|\Delta t|^{\frac{m+\alpha}{2}}}\right). \tag{7}$$

Here k and l are some fixed integers such that

$$k>m+\alpha,\quad l>\frac{m+\alpha}{2},\quad \Delta^k_{x,\overline{h}}u=\Delta_{x,\overline{h}}\big(\Delta^{k-1}_{x,\overline{h}}u\big),\quad \Delta_{x,\overline{h}}u=u(x+\overline{h},t)-u(x,t)$$

and analogously

$$\Delta^l_{t,\Delta t}u=\Delta_{t,\Delta t}\big(\Delta^{l-1}_{t,\Delta t}u\big),\quad \Delta_{t,\Delta t}u=u(x,t+\Delta t)-u(x,t).$$

The above relations can be written also in a more concise way. Denote $\overline{H}=(\overline{h},\Delta t)$,

$$\|\overline{H}\|\equiv|\overline{h}\,|+|\Delta t|^{\frac{1}{2}},$$

and denote $\Delta_{\overline{H}}u(x,t)=u(x+\overline{h},t+\Delta t)-u(x,t)$. Then (7) is equivalent to

$$\langle u\rangle^{(m+\alpha)}_{\overline{\Omega}_T}\simeq C_k\sup_{(x,t),(x,t)+\overline{H}\in\overline{\Omega}_T}\frac{\left|\Delta^k_{\overline{H}}u(x,t)\right|}{\left\|\overline{H}\right\|^{m+\alpha}}. \tag{8}$$

We also use for functions $u(x)$ or $u(x,t)$ spaces $L_p(\Omega)$ or $L_p(\Omega_T)$ respectively with the norms $\|u\|_{p,\Omega}$ and $\|u\|_{p,\Omega_T}$ correspondingly, $p>1$.

For the spaces $C^{m+\alpha}(\overline{\Omega})=C^l(\overline{\Omega})$ with noninteger $l=m+\alpha$ and also with integer $l\geqslant 0$ we have the following interpolation inequalities (see, for example, [2])

$$|u|^{(l)}_{\overline{\Omega}}\leqslant C\left(|u|^{(l_2)}_{\overline{\Omega}}\right)^{\omega}\left(|u|^{(l_1)}_{\overline{\Omega}}\right)^{1-\omega},\quad \omega=\frac{l-l_1}{l_2-l_1},\quad l_1<l<l_2, \tag{9}$$

and analogous inequalities for anisotropic spaces $C^{l,\frac{l}{2}}(\overline{\Omega}_T)$

$$|u|^{(l)}_{\overline{\Omega}_T}\leqslant C\left(|u|^{(l_2)}_{\overline{\Omega}_T}\right)^{\omega}\left(|u|^{(l_1)}_{\overline{\Omega}_T}\right)^{1-\omega},\quad \omega=\frac{l-l_1}{l_2-l_1},\quad l_1<l<l_2, \tag{10}$$

where l_1, l_2 may be either integer or noninteger.

The further content of the paper is as follows. In the next section of the paper we prove some interpolation inequalities for functions from the Hölder spaces (in the case of unbounded domain we need the intersection of the Hölder and the Lesbesgue spaces). And in the last third section we apply these inequalities to a priori estimates and solvability of a model (just for simplicity) problem for partial differential equations.

2. Interpolation Inequalities

We start with the following interpolation inequality as the key particular case.

Lemma 1. *Let $l > 0$ be a positive noninteger and let $u(x,t) \in C^{l,\frac{l}{2}}(\overline{\Omega}_T) \cap L_p(\Omega_T)$. Then*

$$|u|^{(0)}_{\Omega_T} \leqslant C\left(|u|^{(l)}_{\Omega_T}\right)^{\omega}\left(\|u\|_{p,\Omega_T}\right)^{1-\omega}, \quad \omega = \frac{N+2}{lp+N+2}. \tag{11}$$

If for the function $u(x,t)$ the following parabolic norm is finite

$$\sup_{0<t<T} \|u(\cdot,t)\|_{p,\Omega} < \infty, \tag{12}$$

then

$$|u|^{(0)}_{\Omega_T} \leqslant C\left(|u|^{(l)}_{\Omega_T}\right)^{\sigma}\left(\sup_{0<t<T}\|u(\cdot,t)\|_{p,\Omega}\right)^{1-\sigma}, \quad \sigma = \frac{N}{lp+N}. \tag{13}$$

$\lhd$ We are going to use relation (8). Let first $\Omega = R^N$, $T = \infty$. Let $x, y \in R^N$, x is fixed, $t, \tau > 0$, t is fixed, $\overline{h} = y - x$, $\Delta t = t - \tau$, $\overline{H} = (\overline{h}, \Delta t)$, k is an integer, $k > l$, $\varepsilon > 0$. Represent $u(x,t)$ in the form

$$u(x,t) = \Delta^k_{\overline{H}} u + \sum_{i=1}^{k} C_i u\big(x + i\overline{h}, t + i\Delta t\big) = \frac{\Delta^k_{\overline{H}} u}{\|\overline{H}\|^l}\,\|\overline{H}\|^l + \sum_{i=1}^{k} C_i u\big(x + i\overline{h}, t + i\Delta t\big), \tag{14}$$

where C_i, $i = 1, \ldots, k$, are some integers, depending only on k. From this we obtain

$$|u(x,t)| \leqslant C\langle u\rangle^{(l)}_{R^N \times R^1_+}\|\overline{H}\|^l + \sum_{i=1}^{k} c_i\big|u\big(x + i\overline{h}, t + i\Delta t\big)\big| \tag{15}$$

with some constants c_i, $R^1_+ = \{t \geqslant 0\}$.

Raising this inequality to the power $p > 1$ and applying the inequality

$$\left(\sum_{i=1}^{N} a_i\right)^p \leqslant N^p \sum_{i=1}^{N} a_i^p, \quad N \in \mathbb{N},\ a_i > 0,\ i = 1, \ldots, N,\ p > 0,$$

we obtain

$$|u(x,t)|^p \leqslant C\left(\langle u\rangle^{(l)}_{R^N \times R^1_+}\right)^p \|\overline{H}\|^{pl} + C\sum_{i=1}^{k} |u|^p\big(x + i\overline{h}, t + i\Delta t\big). \tag{16}$$

Integrating in (y,τ) over the cylinder $Q_\varepsilon(x,t) = \{(y,\tau) : |y - x| \leqslant \varepsilon,\ t \leqslant \tau \leqslant t + \varepsilon^2\}$, we get

$$C\varepsilon^{N+2}\,|u(x,t)|^p \leqslant C\left(\langle u\rangle^{(l)}_{R^N \times R^1_+}\right)^p \varepsilon^{N+2+pl} + C_i\sum_{i=1}^{k}\int\limits_{Q_{i\varepsilon}(x,t)} |u|^p(z,\theta)\,dz\,d\theta, \tag{17}$$

where for each i in the sum we made the change of the variables $z = x + i(y - x)$, $\theta = t + i(\tau - t)$ and we took into account that for each i we have $(y - x) = (z - x)/i$. If the norm in (12) is finite, we can integrate just in y over the ball $B_\varepsilon(x,t) = \{y : |y - x| \leqslant \varepsilon\}$ and obtain

$$C\varepsilon^N|u(x,t)|^p \leqslant C\left(\langle u\rangle^{(l)}_{R^N \times R^1_+}\right)^p \varepsilon^{N+pl} + C_i\sum_{i=1}^{k}\int\limits_{B_{i\varepsilon}(x,t+i\Delta t)} |u|^p(z, t + i\Delta t)\,dz. \tag{18}$$

Estimate now the integrals over $Q_{i\varepsilon}(x,t)$ in (17) by the integral over $R^N \times R^1_+$, divide both sides of (17) by $C\varepsilon^{N+2}$ and take the roots of power p from the terms of this relation. As a result we obtain

$$|u(x,t)| \leqslant C\varepsilon^l \langle u\rangle^{(l)}_{R^N\times R^1_+} + C\varepsilon^{-\frac{N+2}{p}} \|u\|_{p,R^N\times R^1_+},$$

or, taking supremum over $(x,t) \in R^N \times R^1_+$,

$$|u|^{(0)}_{R^N\times R^1_+} \leqslant C\varepsilon^l \langle u\rangle^{(l)}_{R^N\times R^1_+} + C\varepsilon^{-\frac{N+2}{p}} \|u\|_{p,R^N\times R^1_+}. \tag{19}$$

Optimizing this inequality with respect to $\varepsilon > 0$, or just taking

$$\varepsilon = \left(\frac{\|u\|_{p,R^N\times R^1_+}}{\langle u\rangle^{(l)}_{R^N\times R^1_+}}\right)^{\frac{p}{pl+N+2}}$$

with $\langle u\rangle^{(l)}_{R^N\times R^1_+} \neq 0$, we finally obtain

$$|u|^{(0)}_{R^N\times R^1_+} \leqslant C\left(\langle u\rangle^{(l)}_{R^N\times R^1_+}\right)^{\frac{N+2}{pl+N+2}} \left(\|u\|_{p,R^N\times R^1_+}\right)^{\frac{pl}{pl+N+2}} \tag{20}$$

that is exactly inequality (11). If now $\langle u\rangle^{(l)}_{R^N\times R^1_+} = 0$, then from (19) after letting $\varepsilon \to \infty$ it follows that $|u|^{(0)}_{R^N\times R^1_+} = 0$ and so (20) is valid in this case also.

If the norm in (12) is finite, completely analogously to (20), we have subsequently from (18)

$$|u(x,t)| \leqslant C\varepsilon^l \langle u\rangle^{(l)}_{R^N\times R^1_+} + C\varepsilon^{-\frac{N}{p}} \sup_{t\in R^1_+} \|u(\cdot,t)\|_{p,R^N},$$

$$|u|^{(0)}_{R^N\times R^1_+} \leqslant C\varepsilon^l \langle u\rangle^{(l)}_{R^N\times R^1_+} + C\varepsilon^{-\frac{N}{p}} \sup_{t\in R^1_+} \|u(\cdot,t)\|_{p,R^N},$$

and, optimizing this inequality with respect to $\varepsilon > 0$,

$$|u|^{(0)}_{R^N\times R^1_+} \leqslant C\left(\langle u\rangle^{(l)}_{R^N\times R^1_+}\right)^{\frac{N}{pl+N}} \left(\sup_{t\in R^1_+} \|u(\cdot,t)\|_{p,R^N}\right)^{\frac{pl}{pl+N}}$$

that is exactly inequality (13).

Now in the case of general smooth domain $\Omega \neq R^N$ and $T < \infty$ the lemma follows by an extension of a given function to $R^N \times R^1_+$ with the preserving of the corresponding norms up to a multiple constant (the way of such extension for smooth domains is described in, for example, [14, Ch. IV]). The lemma is proved. ⊳

Now we can easily get the following more general assertion.

Theorem 1. *Let l be any positive number and let $l_2 > l$ be a positive noninteger. Let also $u(x,t) \in C^{l_2,\frac{l_2}{2}}(\overline{\Omega}_T) \cap L_p(\Omega_T)$. Then*

$$|u|^{(l)}_{\overline{\Omega}_T} \leqslant C\left(|u|^{(l_2)}_{\overline{\Omega}_T}\right)^{\omega} \left(\|u\|_{p,\overline{\Omega}_T}\right)^{1-\omega}, \quad \omega = \frac{pl+N+2}{pl_2+N+2}. \tag{21}$$

If the parabolic norm

$$\sup_{0<t<T} \|u(\cdot,t)\|_{p,\Omega} < \infty$$

is finite, then

$$|u|^{(l)}_{\overline{\Omega}_T} \leqslant C\left(|u|^{(l_2)}_{\overline{\Omega}_T}\right)^{\sigma}\left(\sup_{0<t<T}\|u(\cdot,t)\|_{p,\Omega}\right)^{1-\sigma}, \quad \sigma = \frac{pl+N}{pl_2+N}\,. \tag{22}$$

$\lhd$ From (10) with $l_1 = 0$ we have

$$|u|^{(l)}_{\overline{\Omega}_T} \leqslant C\left(|u|^{(l_2)}_{\overline{\Omega}_T}\right)^{\frac{l}{l_2}}\left(|u|^{(0)}_{\overline{\Omega}_T}\right)^{\frac{l_2-l}{l_2}}.$$

At the same time, from (11) it follows that

$$|u|^{(0)}_{\overline{\Omega}_T} \leqslant C\left(|u|^{(l_2)}_{\overline{\Omega}_T}\right)^{\omega}\left(\|u\|_{p,\Omega_T}\right)^{1-\omega}, \quad \omega = \frac{N+2}{l_2p+N+2}\,.$$

Substituting this estimate for $|u|^{(0)}_{\overline{\Omega}_T}$ in the previous inequality, we obtain

$$\begin{aligned}|u|^{(l)}_{\overline{\Omega}_T} &\leqslant C\left(|u|^{(l_2)}_{\overline{\Omega}_T}\right)^{\frac{l}{l_2}}\left[\left(|u|^{(l_2)}_{\overline{\Omega}_T}\right)^{\frac{N+2}{pl_2+N+2}}\left(\|u\|_{\overline{\Omega}_T}\right)^{\frac{pl_2}{pl_2+N+2}}\right]^{\frac{l_2-l}{l_2}}\\ &= C\left(|u|^{(l_2)}_{\overline{\Omega}_T}\right)^{\frac{pl+N+2}{pl_2+N+2}}\left(\|u\|_{\overline{\Omega}_T}\right)^{\frac{p(l_2-l)}{pl_2+N+2}}\end{aligned}$$

that is (21). Inequality (22) is completely analogous on the base of (13). The Theorem is proved. $\rhd$

By exactly the same arguments we have also an assertion for isotropic the Hölder spaces in an “elliptic” case.

Theorem 2. *Let l be any positive number and let $l_2 > l$ be a positive noninteger. Let also $u(x) \in C^{l_2}(\overline{\Omega}) \cap L_p(\Omega)$. Then*

$$|u|^{(l)}_{\overline{\Omega}} \leqslant C\left(|u|^{(l_2)}_{\overline{\Omega}}\right)^{\omega}\left(\|u\|_{p,\overline{\Omega}}\right)^{1-\omega}, \quad \omega = \frac{pl+N}{pl_2+N}\,. \tag{23}$$

In the next section we give some simple application to an initial boundary value problem for a quasilinear parabolic equation, mostly to illustrate the idea of applications.

3. Solvability of a Quasilinear Initial Boundary Value Problem

Consider the following initial boundary value problem in a bounded domain $\overline{\Omega}_T$ for an unknown function $u(x,t)$

$$\frac{\partial u}{\partial t} - \triangle u + |u|^{q-2}u = f(x,t), \quad (x,t) \in \Omega_T, \tag{24}$$

$$u(x,t) = 0, \quad x \in \partial\Omega, \tag{25}$$

$$u(x,0) = u_0(x), \quad x \in \overline{\Omega}. \tag{26}$$

Here $f(x,t)$ is a given function, $u_0(x)$ is a given initial datum and we suppose that

$$f(x,t) \in C^{\alpha,\frac{\alpha}{2}}\big(\overline{\Omega}_T\big), \quad u_0(x) \in C^{2+\alpha}\big(\overline{\Omega}\,\big), \quad 2 \leqslant q < 2 + \frac{4}{N}. \tag{27}$$

We note again that we consider the case of a bounded domain Ω just for simplicity of estimates. (In the case of an unbounded domain we can consider the data of the problem to be from an appropriate the Lebesgue spaces, besides (27).) We are going to prove carefully only a priori estimate for a solution to problem (24)–(26) in the space $C^{2+\alpha,\frac{2+\alpha}{2}}(\overline{\Omega}_T)$. The existence and uniqueness of the solution can be proved after this in a more-less standard (nowadays) way (see, for example, [2, Ch. 7]), about the quasilinear parabolic equations. Namely, we have the following assertion.

Theorem 3. *Let* $\alpha \in (0,1)$. *Let, further, a function* $u(x,t) \in C^{2+\alpha,\frac{2+\alpha}{2}}(\overline{\Omega}_T)$ *satisfy problem* (24)–(26). *Let also*

$$A = \frac{q\alpha + (q-1)(N+2)}{q(2+\alpha)+N+2}, \quad B = \frac{q\left[(q-2)(2+\alpha)+2\right]}{q(2+\alpha)+N+2}. \tag{28}$$

There exists a constant $C > 0$, *which does not depend on* $f(x,t)$ *and* $u_0(x)$, *with*

$$|u|^{(2+\alpha)}_{\overline{\Omega}_T} \leqslant C\left(|f|^{(\alpha)}_{\overline{\Omega}_T} + |u_0|^{(2+\alpha)}_{\overline{\Omega}}\right) + C\left(\int_0^T\int_\Omega f^2(x,t)\,dxdt + \int_\Omega u_0^2(x)\,dx\right)^{\frac{B}{q(1-A)}}. \tag{29}$$

◁ First we get some integral estimates for $u(x,t)$. Multiply equation (24) by $u(x,t)$ and integrate over Ω_T. After integrating by parts in the first two terms of the equation with taking into account the boundary and the initial conditions we get

$$\frac{1}{2}\int_\Omega u^2(x,T)dx + \int_0^T\int_\Omega |\nabla u|^2\,dxdt + \int_0^T\int_\Omega |u|^q\,dxdt = \int_0^T\int_\Omega f(x,t)u(x,t)dxdt + \frac{1}{2}\int_\Omega u_0^2(x)dx. \tag{30}$$

For the first term in the right hand side of (30) we have

$$\left|\int_0^T\int_\Omega f(x,t)u(x,t)\,dxdt\right| \leqslant \|f\|_{2,\Omega_T}\|u\|_{2,\Omega_T} \leqslant \varepsilon\|u\|^2_{2,\Omega_T} + \frac{1}{\varepsilon}\|f\|^2_{2,\Omega_T}$$

$$= \varepsilon\int_0^T\int_\Omega u^2(x,t)\,dxdt + \frac{1}{\varepsilon}\int_0^T\int_\Omega f^2(x,t)\,dxdt \leqslant \varepsilon C_\Omega\int_0^T\int_\Omega |\nabla u|^2(x,t)\,dxdt + \frac{1}{\varepsilon}\int_0^T\int_\Omega f^2(x,t)\,dxdt,$$

where we took advantage first of the Hölder inequality, then of the Cauchy inequality with ε, and then of the Poincare inequality. Taking into account this estimate, choosing ε such that $\varepsilon C_\Omega = 1/2$, and absorbing the term with ε into the left hand side of (30), we obtain

$$\int_\Omega u^2(x,T)\,dx + \int_0^T\int_\Omega |\nabla u|^2 dxdt + \int_0^T\int_\Omega |u|^q dxdt \leqslant C\left(\int_0^T\int_\Omega f^2(x,t)\,dxdt + \int_\Omega u_0^2(x)\,dx\right). \tag{31}$$

Thus, in particular, we get the following estimate

$$\|u\|_{q,\Omega_T} \leqslant C\left(\int_0^T\int_\Omega f^2(x,t)\,dxdt + \int_\Omega u_0^2(x)\,dx\right)^{\frac{1}{q}}. \tag{32}$$

Now we are ready to obtain a priori estimate for a solution to (24)–(26) from the space $C^{2+\alpha,\frac{2+\alpha}{2}}(\overline{\Omega}_T)$. Let us apply inequality (21) to estimate the norm of the term $|u|^{q-2}u$ in the space $C^{\alpha,\frac{\alpha}{2}}(\overline{\Omega}_T)$. Note first that by elementary considerations we have

$$\big||u|^{q-2}u\big|^{(\alpha)}_{\Omega_T} \leqslant C\left(|u|^{(0)}_{\Omega_T}\right)^{q-2}|u|^{(\alpha)}_{\Omega_T}. \tag{33}$$

Use (21) with $l_2 = 2+\alpha$, $l_1 = 0$, $p = q$ to obtain

$$|u|^{(0)}_{\Omega_T} \leqslant C\left(|u|^{(2+\alpha)}_{\overline{\Omega}_T}\right)^{\omega_0}\left(\|u\|_{q,\overline{\Omega}_T}\right)^{1-\omega_0}, \quad \omega_0 = \frac{N+2}{q(2+\alpha)+N+2}\,. \tag{34}$$

Then use (21) with $l_2 = 2+\alpha$, $l_1 = \alpha$, $p = q$ to obtain also

$$|u|^{(\alpha)}_{\Omega_T} \leqslant C\left(|u|^{(2+\alpha)}_{\overline{\Omega}_T}\right)^{\omega_\alpha}\left(\|u\|_{q,\overline{\Omega}_T}\right)^{1-\omega_\alpha}, \quad \omega_\alpha = \frac{q\alpha+N+2}{q(2+\alpha)+N+2}\,. \tag{35}$$

From (33)–(35) it follows that

$$\big||u|^{q-2}u\big|^{(\alpha)}_{\Omega_T} \leqslant C\left(|u|^{(2+\alpha)}_{\overline{\Omega}_T}\right)^{A}\left(\|u\|_{q,\overline{\Omega}_T}\right)^{B}, \tag{36}$$

where

$$A = (q-2)\omega_0 + \omega_\alpha, \quad B = (q-2)(1-\omega_0) + 1 - \omega_\alpha \tag{37}$$

are defined in (28). On the base of the condition on q in (27) we have $A < 1$. So, applying to (36) the Young inequality with ε, we obtain

$$\big||u|^{q-2}u\big|^{(\alpha)}_{\Omega_T} \leqslant \varepsilon|u|^{(2+\alpha)}_{\overline{\Omega}_T} + C^{\frac{1}{1-A}}\,\varepsilon^{-\frac{A}{1-A}}\left(\|u\|_{q,\overline{\Omega}_T}\right)^{\frac{B}{1-A}}. \tag{38}$$

Moving now the term $|u|^{q-2}u$ to the right hand side of equation (24) we represent it in the form

$$\frac{\partial u}{\partial t} - \triangle u = g(x) \equiv f(x,t) - |u|^{q-2}u, \quad (x,t) \in \Omega_T. \tag{39}$$

Now we use the well known estimate in Hölder spaces for a solution to initial-boundary problem (39), (25), (26) (with the given function $|u|^{q-2}u$) to obtain

$$\begin{aligned}
|u|^{(2+\alpha)}_{\overline{\Omega}_T} &\leqslant C\left(|g|^{(\alpha)}_{\overline{\Omega}_T} + |u_0|^{(2+\alpha)}_{\overline{\Omega}}\right) \leqslant C\left(|f|^{(\alpha)}_{\overline{\Omega}_T} + |u_0|^{(2+\alpha)}_{\overline{\Omega}} + \big||u|^{q-2}u\big|^{(\alpha)}_{\Omega_T}\right)\\
&\leqslant C\left(|f|^{(\alpha)}_{\overline{\Omega}_T} + |u_0|^{(2+\alpha)}_{\overline{\Omega}}\right) + C\varepsilon|u|^{(2+\alpha)}_{\overline{\Omega}_T} + C\varepsilon^{-\frac{A}{1-A}}\left(\|u\|_{q,\overline{\Omega}_T}\right)^{\frac{B}{1-A}}\\
&\leqslant C\varepsilon|u|^{(2+\alpha)}_{\overline{\Omega}_T} + C\left(|f|^{(\alpha)}_{\overline{\Omega}_T} + |u_0|^{(2+\alpha)}_{\overline{\Omega}}\right) + C\varepsilon^{-\frac{A}{1-A}}\left(\int_0^T\!\!\int_\Omega f^2(x,t)dxdt + \int_\Omega u_0^2(x)dx\right)^{\frac{B}{q(1-A)}},
\end{aligned}$$

where we took advantage of (38), and then (32). Absorbing now the first term with sufficiently small ε ($C\varepsilon = 1/2$) into the left hand side of the last inequality, we arrive at the estimate

$$|u|^{(2+\alpha)}_{\overline{\Omega}_T} \leqslant C\left(|f|^{(\alpha)}_{\overline{\Omega}_T} + |u_0|^{(2+\alpha)}_{\overline{\Omega}}\right) + C\left(\int_0^T\!\!\int_\Omega f^2(x,t)\,dxdt + \int_\Omega u_0^2(x)\,dx\right)^{\frac{B}{q(1-A)}}$$

that is at (29). Theorem is proved. $\rhd$

Note that, of cause, the estimate of the kind (29) can be obtained in some other ways. For example, starting with estimate (31), one can consider problem (24)–(26) in the Sobolev spaces first (under even weaker restrictions on the exponent q). And then one can use some bootstrap arguments to gradually raise up the smoothness of the solution with some corresponding estimates. For example, we can first consider the problem in the space $W^{2,1}_{\frac{q}{q-1}}(\Omega_T)$ when the term $|u|^{q-2}u \in L_{\frac{q}{q-1}}$. Well known results on parabolic equations in the Sobolev spaces give us the solution from $W^{2,1}_{\frac{q}{q-1}}(\Omega_T)$. Then the Sobolev embedding gives us that $u \in L_{p_1}(\Omega_T)$ and depending on q and n it may occur $p_1 > q$. Now we can repeat the considerations in the space $W^{2,1}_{\frac{p_1}{q-1}}(\Omega_T)$ to obtain $u \in L_{p_2}(\Omega_T)$ with $p_2 > p_1$. And so on till by embedding $u \in C^{\varepsilon}(\Omega_T)$ with some $\varepsilon > 0$.

Our goal was just to demonstrate how easy it is to apply the interpolation inequalities from section 2 to a priori estimates of solutions to nonlinear PDE in smooth classes of functions.

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О НЕКОТОРЫХ ИНТЕРПОЛЯЦИОННЫХ НЕРАВЕНСТВАХ, ПОЛУЧЕННЫХ О. А. ЛАДЫЖЕНСКОЙ, И НЕЛИНЕЙНЫХ УРАВНЕНИЯХ В ЧАСТНЫХ ПРОИЗВОДНЫХ

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Аннотация. В статье рассмотрены некоторые мультипликативные интерполяционные неравенства между пространствами Гельдера и Лебега. Мультипликативные интерполяционные неравенства типа Гальярдо — Ниренберга широко используются в исследованиях по дифференциальным уравнениям в частных производных. Ранее были доказаны и применены несколько типов таких неравенств, включающих норму (полунорму) Гельдера. Настоящая статья обобщает имеющиеся результаты на случай анизотропных «параболических» пространств, предлагая простое доказательство, основанное на идее О. А. Ладыженской. В работе приводится применение такого неравенства типа Гальярдо — Ниренберга с нормой Гельдера. Используя более слабую интегральную оценку, это неравенство позволяет легко получить априорную оценку решения квазилинейной параболической задачи в гладких классах Гельдера. На основании этой априорной оценки устанавливается существование решения этой задачи. Для доказательства мультипликативного неравенства типа Гальярдо — Ниренберга с нормой Гельдера используется эквивалентная нормировка пространств Гельдера высоких порядков в терминах поведения конечных разностей высокого порядка. Ключевой технический прием заключается в представлении значения функции $u(x,t)$ в произвольной точке (x,t) в терминах ее конечной разности высокого порядка в этой точке, а также добавочной суммы значений функции в соседних точках. После этого производится интегрирование по соседним точкам по шарам $B_r((x,t))$ малого радиуса r с центром в (x,t). Оценивая конечную разность через полунорму Гельдера, мы приходим к аддитивному неравенству с параметром r, которое включает полунорму Гельдера и интегральную норму. Наконец, оптимизируя полученное аддитивное неравенство по параметру r, приходим непосредственно к мультипликативному неравенству, включающему нормы Гельдера и Лебега.

Ключевые слова: интерполяционные неравенства, априорные оценки, нелинейные дифференциальные уравнения.

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