Владикавказский математический журнал
2025, Том 27, Выпуск 3, С. 50–59

УДК 517
DOI 10.46698/o8833-7719-4418-f

STOILOW FACTORIZATION OF THE HEISENBERG GROUP[#]

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Abstract. In this article we study the properties of quasiconformal mappings on the Heisenberg group $\mathbb{H}^1$ and consider the definition of quasiconformal mappings in terms of the Beltrami equation. In particular, we obtain an explicit expression for the Beltrami coefficient for the composition of two quasiconformal mappings and we prove an analogue of the Stoilow factorization theorem on the plane. Namely, if the Beltrami coefficients of two quasiconformal mappings are equal almost everywhere, then there exists a conformal mapping such that by acting on one of the given quasiconformal mappings from the left, we obtain another given mapping. As an application of these results on the Heisenberg group $\mathbb{H}^1$ we compute the Beltrami coefficients of some quasiconformal mappings and we prove a theorem on the images of quasi–Brownian motions. In specific examples we demonstrate the invariance of the Beltrami coefficient under the action of the composition of a conformal function on the corresponding left mapping. Using the Stoilow factorization on the Heisenberg group, we show that if two quasi–Brownian motions have the corresponding Beltrami coefficients equal almost everywhere, then their trajectories are equivalent only if the conformal map in the Stoilov factorization is a map obtained from a composition of translations, rotations and dilations.

Keywords: Heisenberg group, Stoilow factorization, quasiconformal mappings, Beltrami system, Brownian motion.

AMS Subject Classification: 53C17, 34C05.

For citation: *Dorokhin, D. K.* Stoilow Factorization of the Heisenberg Group, *Vladikavkaz Math. J.*, 2025, vol. 27, no. 3, pp. 50–59. DOI: 10.46698/o8833-7719-4418-f.

1. Introduction

It is known, see, for example, [1], that $W^{1,2}_{loc}$-homeomorphism $f : \Omega \to \Omega'$, $\Omega, \Omega' \subset \mathbb{C}$, is K-quasiconformal if and only if, when

$$\frac{\partial f}{\partial \overline{z}}(z) = \mu \frac{\partial f}{\partial z}(z) \quad \text{a. e. } z \in \Omega, \tag{1}$$

where $\mu = \mu(z)$ (*the Beltrami coefficient*) is a bounded measurable function satisfying

$$\|\mu\|_\infty \leqslant \frac{K-1}{K+1} < 1.$$

The equation (1) is called *the Beltrami equation.* Next theorem characterizes all solutions of the Beltrami equation.

[#] The work was supported by the Russian Science Foundation, project № 24-21-00319.

Theorem 1 (Stoilow Factorization). *Let a homeomorphism* $f(z) \in W^{1,2}_{\mathrm{loc}}$ *be a solution of the Beltrami equation* (1) *and* $|\mu(z)| \leqslant k < 1$. *Also let* $g(z) \in W^{1,2}_{\mathrm{loc}}$ *be any other solution of the Beltrami equation* (1). *Then there exists a holomorphic function* $\Phi : \Omega' \to \mathbb{C}$ *such that*

$$g(z) = \Phi(f(z)), \quad z \in \Omega. \tag{2}$$

Conversely, if Φ *is holomorphic on* Ω', *then the composition* $\Phi \circ f$ *is* $W^{1,2}_{\mathrm{loc}}$*-solution of the equation* (1) *on* Ω.

Initially, S. Stoilow in the paper [2] showed that if there is a continuous, open mapping f between two Riemann surfaces S and S' for which the preimage of any point is a totally disconnected set, then there exists the Riemann surface $\widetilde{S}$ and a homeomorphism $h : S \to \widetilde{S}$ such that $f \circ h^{-1} : \widetilde{S} \to S'$ is a holomorphic mapping. Theorem has significant generalizations in 2D analysis: every open and discrete map h is topologically equivalent to some analytic function, so $h = \varphi \circ f$, where f is some homeomorphism, φ is some holomorphic function (mapping $h : \Omega \to \mathbb{C}$ is called discrete, if set $h^{-1}(w)$ is discrete for every $w \in \mathbb{C}$). Y. G. Reshetnyak proved [3] that every mappings with bounded distortion are open and discrete; on the other hand, the quasiconformal mappings are homeomorphic mapping with bounded distortion.

The Beltrami equation has importance for isometrical coordinates on the 2-dimensional manifolds and also in the calculus of variations in minimizing the energy functional for homeomorphisms acted from a unit disk to defined domain [1].

Quasiconformal maps on non-Riemannian structures were first considered by G. D. Mostov because of the classification of metric spaces of constant negative curvature [4]. To prove the rigidity theorem, G. D. Mostow needed quasiconformal transformations of the ideal boundary of some symmetric space [5]. M. L. Gromov, using the Gromov–Hausdorff convergence, proved [6–8] that the geometry of such an ideal boundary is modeled by a nilpotent group with the Carnot–Caratheodory metric. This was one of the incentives for studying quasiconformal maps on the Carnot groups and more general Carnot spaces [8, 9]. In the paper [10] P. Pansu introduced the concept of differentiability of mappings “in terms” of the Carnot–Caratheodory metric ($\mathscr{P}$-differentiability). Using the concept of $\mathscr{P}$-differentiability, A. Koranyi and H. M. Reimann [11] systematized analytical methods for studying mappings on the Heisenberg groups $\mathbb{H}^n$, assuming a priori $\mathscr{P}$-differentiability of mappings almost everywhere. The analytical apparatus that allows one to develop the theory of quasiconformal mappings on the Carnot groups under minimal assumptions was developed by S. K. Vodopyanov and his students, see [12–16]. A. Koranyi and H. M. Reimann [11, 17] showed that on the Heisenberg groups quasiconformal mappings can be defined in terms of systems of differential equations similar to the classical Beltrami equations on the plane. It should be noted separately that in the Euclidean case the Beltrami equations exist only in dimension 2; on the Heisenberg groups $\mathbb{H}^n$ an analogue of the Beltrami equation exists in all dimensions [11, 17]. For strictly pseudoconvex hypersurfaces the theory of the Beltrami equations was constructed in [18].

In our paper, we proved an analogue of the Stoilow factorization theorem on the first Heisenberg group $\mathbb{H}^1$ (Theorem 4). Our proof is based on the use of a formula for the Beltrami coefficient of a composition of quasiconformal mappings and an analogue of the Liouville theorem proved by D. V. Isangulova in the paper [19]. To do this, we derive the notation forms of the corresponding differential operators from the composition of quasiconformal mappings (Lemma 1), using which we calculate the corresponding Beltrami coefficient for compositions of quasiconformal mappings (Lemma 2). The last section of the paper is devoted to the following problem: let two quasiconformal mappings with a common Beltrami coefficient be given; will the images of the corresponding Brownian motions then be equivalent, that is, will these processes be expressed through each other by means of a time change (preserve their trajectories)? We have proven the following

Theorem 2. *Let N_t and M_t be quasi-Brownian motions on the Heisenberg group $\mathbb{H}^1$ with corresponding quasiconformal maps f and g whose Beltrami coefficients are equal almost everywhere. Then, if the map $g \circ f^{-1}$ is a composition of dilations, rotations, and translations on the Heisenberg group $\mathbb{H}^1$, then there exists a time change $a(t)$ such that almost surely $N_t = M_{a(t)}$.*

2. Basic Definitions and Known Theorems

The first Heisenberg group $\mathbb{H}^1$ [11, 17, 19, 20] defined in the standard Euclidean space $\mathbb{R}^3$ with the coordinate system (x, y, t) induced by the coordinate frame (O, e_1, e_2, e_3), using the following table of commutators

$$\begin{cases} [e_1, e_2] = -4e_3, \\ [e_1, e_3] = [e_2, e_3] = 0. \end{cases} \tag{3}$$

Using the Baker–Campbell–Hausdorff formula and the table (3), we obtain an analytical expression of the left translation of $w_1 * w_2$ an arbitrary element $w_2 = (x_2, y_2, t_2) \in \mathbb{H}^1$ and an other arbitrary element $w_1 = (x_1, y_1, t_1) \in \mathbb{H}^1_\alpha$:

$$w_1 * w_2 = (x_1 + x_2,\ y_1 + y_2,\ t_1 + t_2 - 2x_1y_2 + 2y_1x_2). \tag{4}$$

Using (4), we obtain expressions for the basis of left-invariant vector fields (the Jacobi basis [20]) of the Lie algebra V of the group $\mathbb{H}^1$ at each point (x, y, t):

$$X = \frac{\partial}{\partial x} + 2y\,\frac{\partial}{\partial t}, \quad Y = \frac{\partial}{\partial y} - 2x\,\frac{\partial}{\partial t}, \quad T = \frac{\partial}{\partial t}.$$

Left-invariant vector fields X, Y are horizontal. We denote by V_1 the tangent bundle induced by horizontal vector fields X, Y. We have $V = V_1 \oplus V_2$, where V_2 as tangent subbundle, induced "vertical" field T [11, 20].

We will also use the representation of the Heisenberg group. $\mathbb{H}^1$ as $(z, t) \in \mathbb{C} \times \mathbb{R}$ [11, 17]. Then the group operation is defined as

$$(z_1, t_1) * (z_2, t_2) = \big(z_1 + z_2, t_1 + t_2 - 2\operatorname{Im}(z_1 \cdot \bar{z}_2)\big), \tag{5}$$

where $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$.

Action is a one-parameter stretching group $\delta_s : \mathbb{H}^1 \to \mathbb{H}^1$, $s > 0$, set by the law

$$\delta_s(z, t) = \big(sz, s^2 t\big).$$

Homogeneous norm [17] (the Koranyi norm) $\rho(z, t) = (|z|^4 + t^2)^{1/4}$ sets the metric on $\mathbb{H}^1$:

$$\rho\big((z_1, t_1), (z_2, t_2)\big) = \rho\big((z_2, t_2)^{-1} * (z_1, t_1)\big).$$

Definition 1 [19]. Let $\Omega \subset \mathbb{H}^1$ be an open set. Sobolev space $W^{1,q}(\Omega)$, where $1 \leqslant q \leqslant \infty$, consists of locally integrable functions $f : \Omega \to \mathbb{R}$, having weak derivatives Xf and Yf, with norm

$$\|f\|_{W^{1,q}(\Omega)} = \|f\|_{L^q(\Omega)} + \|\nabla_h f\|_{L^q(\Omega)} < \infty,$$

where $\nabla_h f = (Xf, Yf)$ is a horizontal gradient. We will denote $f \in W^{1,q}_{\mathrm{loc}}(\Omega)$, if $f \in W^{1,q}(U)$ for any open set U such that $\overline{U} \subset \Omega$.

Definition 2 [19, 20]. A mapping $f : \Omega \to \mathbb{H}^1$ belongs to $W^{1,q}_{\mathrm{loc}}(\Omega, \mathbb{H}^1)$, if $f \in L^q(\Omega, \mathbb{H}^1)$ and satisfies:

(A) for any $w \in \mathbb{H}$ a function $[f]_w : x \in \Omega \mapsto \rho(f(x), w)$ belongs to $W^{1,q}_{\mathrm{loc}}(\Omega)$;

(B) a family of functions $\nabla_h [f]_w$ has a majority belongs to $L^q_{\mathrm{loc}}(\Omega)$.

Recall that the differential 1-form τ sets the contact structure on a $(2n+1)$-dimensional manifold, if $\tau \wedge (d\tau)^n \neq 0$. On the Heisenberg group $\mathbb{H}^1$ the contact structure is determined by the form

$$\tau = 2xdy - 2ydx + dt. \tag{6}$$

We have $\ker \tau = V_1$.

Definition 3 [17]. A diffeomorphism $f : U \to U'$, $U, U' \subset \mathbb{H}^1$, will be called a contact map if it preserves the contact structure, that is

$$f^*\tau = \lambda\tau, \tag{7}$$

where $\lambda \neq 0$ is some function.

Condition (7) can be written in the following equivalent form

$$\begin{cases} -2f_2Xf_1 + 2f_1Xf_2 + Xf_3 = 0, \\ -2f_2Yf_1 + 2f_1Yf_2 + Yf_3 = 0. \end{cases} \tag{8}$$

Mapping of the Sobolev space $f : U \to U'$, $U, U' \subset \mathbb{H}^1$ are *weak contact*, that is, the conditions (8) are fulfilled for them almost everywhere (see, for example, [19]).

For mappings $f = (f_1, f_2, f_3) : \Omega \to \mathbb{H}^1$ of the Sobolev space $W^{1,q}_{\mathrm{loc}}(\Omega, \mathbb{H}^1)$ we define the formal horizontal differential $D_h f$ as a matrix

$$D_h f = \begin{pmatrix} Xf_1 & Yf_1 \\ Xf_2 & Yf_2 \end{pmatrix}.$$

It follows from general facts, see, for example, [21], that the horizontal differential $D_h f$ generates a linear mapping $Df : V \to V$, called a formal differential, which is a homomorphism preserving the grading of the Lie algebra V. The determinant of the matrix $Df(x)$ will be called the formal Jacobian of the mapping f and denoted by $J_f(x)$.

Definition 4 [19, Definition 1]. The homeomorphism $f : \Omega \to \mathbb{H}^1$ defined on an open set $\Omega \subset \mathbb{H}$, is a quasiconformal mapping if $f \in W^{1,4}_{\mathrm{loc}}(\Omega; \mathbb{H}^1)$, and there is a constant $K \geqslant 1$, such that the inequality

$$\|D_h f(x)\|^4 \leqslant K J_f(x) \tag{9}$$

for almost any $x \in \Omega$. Consider the following vector fields

$$Z = \frac{1}{2}(X - iY) \quad \text{and} \quad \overline{Z} = \frac{1}{2}(X + iY). \tag{10}$$

Definition 5 [11, Definition 2]. The homeomorphism $f = (f_1, f_2, f_3) : \Omega \to \mathbb{H}^1$ is defined on an open set $\Omega \subset \mathbb{H}^1$, is a quasiconformal mapping, if f is absolutely continuous, $\mathscr{P}$-differentiable almost everywhere, preserves orientation and almost everywhere on Ω satisfies the Beltrami equation

$$\overline{Z} f_{\mathrm{I}} = \mu Z f_{\mathrm{I}}, \tag{11}$$

$$\overline{Z}f_{\mathrm{II}} = \mu Z f_{\mathrm{II}}, \tag{12}$$

where $f_{\mathrm{I}} = f_1 + if_2$ and $f_{\mathrm{II}} = f_3 + i|f_{\mathrm{I}}|^2$, μ is some measurable function, such that $\|\mu\|_\infty < 1$.

These two definitions of a quasiconformal mappping are equivalent, see [11, 21].

We have [22]

$$\begin{gathered}
D_h f = \begin{pmatrix} Zf_{\mathrm{I}} & \overline{Z}f_{\mathrm{I}} \\ Z\overline{f_{\mathrm{I}}} & \overline{Zf_{\mathrm{I}}} \end{pmatrix}, \quad \|D_h f\| = |Zf_{\mathrm{I}}| + \left|\overline{Z}f_{\mathrm{I}}\right|, \\
J_f = \det \begin{pmatrix} Zf_{\mathrm{I}} & \overline{Z}f_{\mathrm{I}} & 0 \\ Z\overline{f_{\mathrm{I}}} & \overline{Zf_{\mathrm{I}}} & 0 \\ 0 & 0 & |Zf_{\mathrm{I}}|^2 - \left|\overline{Z}f_{\mathrm{I}}\right|^2 \end{pmatrix} = \left(|Zf_{\mathrm{I}}|^2 - \left|\overline{Z}f_{\mathrm{I}}\right|^2\right)^2.
\end{gathered} \tag{13}$$

Remark 1. If the Beltrami coefficient of the quasiconformal mapping f is zero almost everywhere, then almost everywhere it holds $\overline{Z}f_{\mathrm{I}} = 0$. Then the inequality (9) has the form

$$|Zf_{\mathrm{I}}|^4 \leqslant K|Zf_{\mathrm{I}}|^4. \tag{14}$$

This is true for all $K \geqslant 1$, i. e. f is a 1-quasiconformal (conformal) mapping.

Theorem 3 [19, p. 326]**.** *Any (not necessarily orientation-preserving) 1-quasiconformal map on the Heisenberg group $\mathbb{H}^1$ is represented as a composition of the following type of mappings:*

(1) *Left translation* $\pi_a(x) = a * x$, $a \in \mathbb{H}$;

(2) *Dilation* $\delta_s(x) = (sz, s^2 t)$, $s > 0$;

(3) *Rotation* $\phi_\alpha(x) = (e^{i\alpha}z, t)$, $\alpha \in \mathbb{R}$;

(4) *Inversion* $j(x) = \left(\frac{z}{|z|^2 - it}, \frac{-t}{\rho(x)^4}\right)$;

(5) *Reflection* $\iota(x) = (\overline{z}, t)$.

3. Stoilow Factorization on the Heisenberg Group $\mathbb{H}^1$

Lemma 1. *Let $\Omega, \Omega', \Omega''$ be domains of $\mathbb{H}^1$, $g : \Omega \to \Omega'$ and $h : \Omega' \to \Omega''$ are quasiconformal mappings. Then*

$$Z(h_{\mathrm{I}} \circ g) = (Zh_{\mathrm{I}} \circ g)Zg_{\mathrm{I}} + \left(\overline{Z}h_{\mathrm{I}} \circ g\right)Z\,\overline{g}_{\mathrm{I}},$$

$$\overline{Z}(h_{\mathrm{I}} \circ g) = (Zh_{\mathrm{I}} \circ g)\overline{Z}g_{\mathrm{I}} + \left(\overline{Z}h_{\mathrm{I}} \circ g\right)\overline{Z}\,\overline{g}_{\mathrm{I}}.$$

◁ We have

$$
\begin{aligned}
Z(h_{\mathrm{I}}\circ g) &= \frac{1}{2}(X-iY)(h_{\mathrm{I}}\circ g) = \frac{1}{2}\big([X(h_1\circ g)+Y(h_2\circ g)]+i\,[X(h_2\circ g)-Y(h_1\circ g)]\big)\\
&= \frac{1}{2}\Big(\big[(h_{1x}\circ g)g_{1x}+(h_{1y}\circ g)g_{2x}+(h_{1t}\circ g)g_{3x}+2y\big((h_{1x}\circ g)g_{1t}+(h_{1y}\circ g)g_{2t}+(h_{1t}\circ g)g_{3t}\big)\big]\\
&\quad+\big[(h_{2x}\circ g)g_{1y}+(h_{2y}\circ g)g_{2y}+(h_{2t}\circ g)g_{3y}-2x\big((h_{2x}\circ g)g_{1t}+(h_{2y}\circ g)g_{2t}+(h_{2t}\circ g)g_{3t}\big)\big]\\
&\quad+i\,\big[(h_{2x}\circ g)g_{1x}+(h_{2y}\circ g)g_{2x}+(h_{2t}\circ g)g_{3x}+2y\big((h_{2x}\circ g)g_{1t}+(h_{2y}\circ g)g_{2t}+(h_{2t}\circ g)g_{3t}\big)\big]\\
&\quad-i\,\big[(h_{1x}\circ g)g_{1y}+(h_{1y}\circ g)g_{2y}+(h_{1t}\circ g)g_{3y}-2x\big((h_{1x}\circ g)g_{1t}+(h_{1y}\circ g)g_{2t}+(h_{1t}\circ g)g_{3t}\big)\big]\Big)\\
&= \big\{\text{combine them } g_{ix}+2yg_{it} \text{ and } g_{iy}-2xg_{it} \text{ into } Xg_1 \text{ and } Yg_1, \text{accordingly, for all } i=1,2,3\big\}\\
&= \frac{1}{2}\big[(h_{1x}\circ g)Xg_1+(h_{1y}\circ g)Xg_2+(h_{1t}\circ g)Xg_3\big]+\big[(h_{2x}\circ g)Yg_1+(h_{2y}\circ g)Yg_2+(h_{2t}\circ g)Yg_3\big]\\
&\quad+i\,\big[(h_{2x}\circ g)Xg_1+(h_{2y}\circ g)Xg_2+(h_{2t}\circ g)Xg_3\big]-i\big[(h_{1x}\circ g)Yg_1+(h_{1y}\circ g)Yg_2+(h_{1t}\circ g)Yg_3\big]\\
&= \big\{\text{by the condition of contact, we will } Xg_3 \text{ and } Yg_3\big\} = \frac{1}{2}\Big(\big[(Xh_1\circ g)Xg_1+(Yh_1\circ g)Xg_2\big]\\
&\quad+\big[(Xh_2\circ g)Yg_1+(Yh_2\circ g)Yg_2\big]+i\big[(Xh_2\circ g)Xg_1+(Yh_2\circ g)Xg_2\big]-i\big[(Xh_1\circ g)Yg_1\\
&\quad+(Yh_1\circ g)Yg_2\big] = \frac{1}{2}\Big(\big[(Xh_{\mathrm{I}}\circ g)Xg_1+(Yh_{\mathrm{I}}\circ g)Xg_2\big]-i\,\big[(Xh_{\mathrm{I}}\circ g)Yg_1+(Yh_{\mathrm{I}}\circ g)Yg_2\big]\Big)\\
&= (Xh_{\mathrm{I}}\circ g)Zg_1+(Yh_{\mathrm{I}}\circ g)Zg_2 = \big(\big(\overline{Z}+Z\big)h_{\mathrm{I}}\circ g\big)Zg_1-i\Big(\big(\overline{Z}-Z\big)\big)h_{\mathrm{I}}\circ g\Big)Zg_2\\
&= (Zh_{\mathrm{I}}\circ g)Zg_{\mathrm{I}}+\big(\overline{Z}h_{\mathrm{I}}\circ g\big)Z\,\overline{g}_{\mathrm{I}}.
\end{aligned}
$$

Similarly:

$$
\begin{aligned}
\overline{Z}(h_{\mathrm{I}}\circ g) &= \frac{1}{2}(X+iY)(h_{\mathrm{I}}\circ g) = \frac{1}{2}\Big(\big[X(h_1\circ g)-Y(h_2\circ g)\big]\\
&\quad+i\,\big[X(h_2\circ g)+Y(h_1\circ g)\big]\Big) = (Zh_{\mathrm{I}}\circ g)\overline{Z}g_{\mathrm{I}}+\big(\overline{Z}h_{\mathrm{I}}\circ g\big)\overline{Z}\,\overline{g}_{\mathrm{I}}. \quad ▷
\end{aligned}
$$

Lemma 2. *Let $f:\Omega\to\Omega_1$ and $g:\Omega\to\Omega_2$ be quasiconformal mappings on the Heisnberg groups $\mathbb{H}^1$ with Beltrami coefficients μ_f and μ_g, accordingly. Then the composition*

$$f\circ g^{-1}:\ \Omega_2\to\Omega_1$$

is an quasiconformal mapping with the Beltrami coefficient

$$\mu_{f\circ g^{-1}}\circ g = \frac{Zg_{\mathrm{I}}}{\overline{Z}g_{\mathrm{I}}}\cdot\frac{\mu_f-\mu_g}{1-\mu_f\,\overline{\mu}_g}. \tag{15}$$

◁ A mapping $f\circ g^{-1}$ is quasiconformal (see, for example, [11]). Let $h=f\circ g^{-1}$, then $f_{\mathrm{I}}=h_{\mathrm{I}}\circ g$. Using the Lemma 1 to Zf_{I} and $\overline{Z}f_{\mathrm{I}}$, we get

$$Zf_{\mathrm{I}} = (Zh_{\mathrm{I}}\circ g)Zg_{\mathrm{I}}+\big(\overline{Z}h_{\mathrm{I}}\circ g\big)Z\,\overline{g}_{\mathrm{I}},\quad \overline{Z}f_{\mathrm{I}} = (Zh_{\mathrm{I}}\circ g)\overline{Z}g_{\mathrm{I}}+\big(\overline{Z}h_{\mathrm{I}}\circ g\big)\overline{Z}\,\overline{g}_{\mathrm{I}}.$$

Then

$$Zh_{\mathrm{I}}\circ g = \frac{Zf_{\mathrm{I}}\overline{Z}\,\overline{g}_{\mathrm{I}}-\overline{Z}f_{\mathrm{I}}Z\,\overline{g}_{\mathrm{I}}}{Zg_{\mathrm{I}}\overline{Z}\,\overline{g}_{\mathrm{I}}-\overline{Z}g_{\mathrm{I}}Z\,\overline{g}_{\mathrm{I}}},\quad \overline{Z}h_{\mathrm{I}}\circ g = -\frac{Zf_{\mathrm{I}}\overline{Z}g_{\mathrm{I}}-\overline{Z}f_{\mathrm{I}}Zg_{\mathrm{I}}}{Zg_{\mathrm{I}}\overline{Z}\,\overline{g}_{\mathrm{I}}-\overline{Z}g_{\mathrm{I}}Z\,\overline{g}_{\mathrm{I}}}.$$

From here we find $\mu_h\circ g$

$$\mu_h\circ g = -\frac{Zf_{\mathrm{I}}\overline{Z}g_{\mathrm{I}}-\overline{Z}f_{\mathrm{I}}Zg_{\mathrm{I}}}{Zf_{\mathrm{I}}\overline{Z}\,\overline{g}_{\mathrm{I}}-\overline{Z}f_{\mathrm{I}}Z\,\overline{g}_{\mathrm{I}}} = \frac{1}{Zf_{\mathrm{I}}\overline{Z}\,\overline{g}_{\mathrm{I}}}\,\frac{\overline{Z}f_{\mathrm{I}}Zg_{\mathrm{I}}-Zf_{\mathrm{I}}\overline{Z}g_{\mathrm{I}}}{1-\mu_f\overline{\mu}_g} = \frac{Zg_{\mathrm{I}}}{\overline{Z}\,\overline{g}_{\mathrm{I}}}\,\frac{\mu_f-\mu_g}{1-\mu_f\overline{\mu}_g}.$$

Note that $1-\mu_f\overline{\mu}_g \neq 0$ almost everywhere, see Definition 5. The product $Zf_{\mathrm{I}}\overline{Z}\overline{g}_{\mathrm{I}}$ is also non-zero almost everywhere. Indeed, if $Zf_{\mathrm{I}}=0$, then using the Beltrami equation for f, we get that $\overline{Z}f_{\mathrm{I}}=0$, from which $J_f=0$, see Remark 1. However, $J_f\neq 0$ is almost everywhere [21, Theorem 4]. Thus, $Zf_{\mathrm{I}}\neq 0$ is almost everywhere. Similar arguments work for $\overline{Z}\overline{g}_{\mathrm{I}}$. ▷

As a result, we obtain the expression for the Beltrami coefficient of the inverse mapping (in the Lemma 2, we need to put $f(x)=x$):

$$\mu_{g^{-1}}\circ g = -\frac{Zg_{\mathrm{I}}}{\overline{Z}\overline{g}_{\mathrm{I}}}\cdot \mu_g. \tag{16}$$

Theorem 4. *Let* $f:\Omega\to\Omega_1$ *and* $g:\Omega\to\Omega_2$ *be quasiconformal mappings on the Heisenberg group* $\mathbb{H}^1$ *with the Beltrami coefficients* μ_f *and* μ_g *accordingly. The following conditions are equivalent:*

(1) $\mu_f=\mu_g$ *almost everythere on* Ω;

(2) *There is an quasiconformal mapping* $h:\Omega_2\to\Omega_1$ *so that* $f=h\circ g$.

◁ (1) ⇒ (2) By the Lemma 2, we consider mappings with almost everywhere equal the Beltrami coefficients. We obtain that the Beltrami coefficient of the function $h=f\circ g^{-1}$ is zero and, therefore, h is a conformal function such that $f=h\circ g$, see Remark 1.

(2) ⇒ (1) Let $f\circ g^{-1}=h$. Then by the Lemma 2 we get

$$0=\mu_{f\circ g^{-1}}\circ g = \frac{Zg_{\mathrm{I}}}{\overline{Z}\overline{g}_{\mathrm{I}}}\cdot\frac{\mu_f-\mu_g}{1-\mu_f\overline{\mu}_g}\,. \tag{17}$$

Therefore, $\mu_f=\mu_g$ almost everywhere. ▷

4. Examples

To demonstrate the results obtained, let us consider a number of examples.

4.1. Invariance of the Beltrami coefficient with respect to the conformal mapping. Now we check that the composition from the left to the conformal mapping does not change the Beltrami coefficient of the mapping. Consider the functions $g=(tz,t^3/3)$, $h=j\circ g=\big(3z/(3t|z|^2-it^2),-3/(9t|z|^4+t^3)\big)$, where j is the inverse of the Heisenberg group. Now μ_g:

$$\mu_g=\frac{\overline{Z}g_{\mathrm{I}}}{Zg_{\mathrm{I}}}=-\frac{izz}{t+i\,|z|^2}\,. \tag{18}$$

We check that μ_g and μ_h are equal almost everywhere:

$$\mu_h=\frac{\overline{Z}h_{\mathrm{I}}}{Zh_{\mathrm{I}}}=\frac{\frac{-9tzz+i3zz(3|z|^2-2it)}{(3t|z|^2-it^2)^2}}{\frac{9t|z|^2-3it^2-9tz\overline{z}-i3\overline{z}z(3|z|^2-2it)}{(3t|z|^2-it^2)^2}}$$
$$=\frac{-3tzz+izz(3|z|^2-2it)}{-it^2-izz(3|z|^2-2it)}=\frac{-izz(3|z|^2+it)}{(t+i|z|^2)(3|z|^2+it)}=-\frac{izz}{t+i|z|^2}\,.$$

4.2. The Beltrami coefficient for the inverse function. We show how the Beltrami coefficient for the inverse mapping is expressed in terms of the Beltrami coefficient of the original mapping. Consider the mappings $h=(tz,t^3/3)$ and $h^{-1}=(z/\sqrt[3]{3t},\sqrt[3]{3t}\,)$. Now μ_h and $\mu_{h^{-1}}$

$$\mu_h=\frac{\overline{Z}h_{\mathrm{I}}}{Zh_{\mathrm{I}}}=-\frac{izz}{t+i|z|^2}\,,\quad \mu_{h^{-1}}=\frac{\overline{Z}h_{\mathrm{I}}^{-1}}{Zh_{\mathrm{I}}^{-1}}=\frac{izz}{3t-i|z|^2}\,.$$

On the other hand, by the Lemma 2

$$\mu_{h^{-1}}=\frac{\overline{Z}(h^{-1}{}_{\mathrm{I}})}{Z(h^{-1}{}_{\mathrm{I}})}=-\left(\frac{Zh_{\mathrm{I}}}{\overline{Zh_{\mathrm{I}}}}\,\mu_h\right)\circ h^{-1}=\left(\frac{t+iz\overline{z}}{t-iz\overline{z}}\cdot\frac{izz}{t+i|z|^2}\right)\circ h^{-1}=\frac{izz}{3t-i|z|^2}\,.$$

4.3. The Beltrami coefficient of the composition of the mappings. Now we show how the Beltrami coefficient for the composition of mappings can be expressed in terms of the Beltrami coefficients of the original mappings. Consider the mappings $h=(2z+\overline{z},3t)$, $g=(tz,t^3/3)$ and $g^{-1}=(z/(3t)^{1/3},(3t)^{1/3})$. Now μ_h, μ_g and $\mu_{g^{-1}}$

$$\mu_h=\frac{\overline{Z}h_{\mathrm{I}}}{Zh_{\mathrm{I}}}=\frac{1}{2}\,,\quad \mu_g=\frac{\overline{Z}g_{\mathrm{I}}}{Zg_{\mathrm{I}}}=-\frac{izz}{t+i|z|^2}\,,\quad \mu_{g^{-1}}=\frac{\overline{Z}g_{\mathrm{I}}^{-1}}{Zg_{\mathrm{I}}^{-1}}=\frac{izz}{3t-i|z|^2}\,.$$

Consider the Beltrami coefficient for $f=h\circ g=(2tz+t\overline{z},-t^3)$:

$$\mu_f=\frac{\overline{Z}f_{\mathrm{I}}}{Zf_{\mathrm{I}}}=\frac{t-iz(2z+\overline{z})}{2t+i\overline{z}(2z+\overline{z})}\,.$$

By the Lemma 2,

$$\mu_{h\circ g}=\frac{\overline{Z}(h_{\mathrm{I}}\circ g)}{Z(h_{\mathrm{I}}\circ g)}=\left(\frac{Zg_{\mathrm{I}}^{-1}}{\overline{Zg_{\mathrm{I}}^{-1}}}\cdot\frac{\mu_h-\mu_{g^{-1}}}{1-\mu_h\overline{\mu_{g^{-1}}}}\right)\circ g=\left(\frac{(3t)^{-\frac{1}{3}}-(3t)^{-\frac{4}{3}}\cdot i\,\overline{z}z}{(3t)^{-\frac{1}{3}}+(3t)^{-\frac{4}{3}}\cdot i\,\overline{z}z}\cdot\frac{\frac{1}{2}-\frac{izz}{3t-i|z|^2}}{1+\frac{1}{2}\frac{i\overline{zz}}{3t+i|z|^2}}\right)\circ g$$

$$=\left(\frac{3t-iz\overline{z}}{3t+iz\overline{z}}\cdot\frac{\frac{3t-i|z|^2-2izz}{3t-i|z|^2}}{\frac{6t+2i|z|^2+i\,\overline{zz}}{3t+i|z|^2}}\right)\circ g=\frac{t^3-it^2|z|^2-2it^2zz}{2t^3+2it^2|z|^2+it^2\overline{zz}}=\frac{t-iz(2z+\overline{z})}{2t+i\overline{z}(2z+\overline{z})}\,.$$

5. Quasi-Brownian Motion on the Heisenberg Group $\mathbb{H}^1$

The property of conformal invariance of the Brownian motion is known: on the plane, the conformal image of the Brownian motion is the Brownian motion with modified time, and in $\mathbb{R}^n$ this is possible only if the mapping is a harmonic morphism [23] (see also [24]). A similar result for mappings on the Heisenberg group $\mathbb{H}^1$ was obtained in [25]. Then the question of describing random processes invariant with respect to quasiconformal mappings of the Heisenberg group is of interest. On the plane, such processes are described in the dissertation [26].

Definition 6 [27]. Let X_t and Y_t be independent standard one-dimensional Brownian motions, and $S_t=2\int_0^t(Y_s dX_s-X_s dY_s)$. The random process $M_t=(X_t,Y_t,S_t)$ will be called the horizontal Brownian motion in the Heisenberg group $\mathbb{H}^1$.

Definition 7. The process A_t will be called the quasi-Brownian motion on the Heisenberg group $\mathbb{H}^1$ if there exists a quasiconformal mapping f such that $f(A_t)$ is the horizontal Brownian motion.

$\lhd$ Proof of Theorem 2. By the definition of the quasi-Brownian motion

$$g(N_t)=\big(g\circ f^{-1}\big)(B_t),$$

where B_t is the Brownian motion on the Heisenberg group. By Theorem 4.1 of [25] the mapping $g\circ f^{-1}$ will preserve the trajectories of the Brownian motion if and only if $g\circ f^{-1}$ is a harmonic morphism on the Heisenberg group, that is, if it is a composition of dilations, rotations, or translations. Note that by Theorem 4 the mapping $g \circ f^{-1}$ is conformal and orientation-preserving. Then we get that $(g \circ f^{-1})(B_t) = \widetilde{B}_{a(t)}$, where $\widetilde{B}_a(t)$ is another independent of B_t Brownian motion with changed time $a(t)$ and $N_t = M_{a(t)}$. $\triangleright$

Acknowledgement. The author expresses his deep gratitude to S. K. Vodopyanov, D. V. Isangulova, N. A. Evseev for attention to the author's results and help.

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Received August 3, 2024

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Владикавказский математический журнал
2025, Том 27, Выпуск 3, С. 50–59

ФАКТОРИЗАЦИЯ СТОИЛОВА НА ГРУППЕ ГЕЙЗЕНБЕРГА

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Аннотация. В данной статье мы исследуем свойства квазиконформных отображений на группе Гейзенберга $\mathbb{H}^1$ и рассматриваем определение квазиконформных отображений через уравнение Бельтрами. В частности, получено явное выражение коэффициента Бельтрами для композиции двух квазиконформных отображений и доказан аналог факторизационной теоремы Стоилова на плоскости. А именно, если коэффициенты Бельтрами двух квазиконформных отображений почти всюду равны, то существует конформное отображение такое, что подействовав им слева на какой-то из данных квазиконформных отображений, мы получим другое заданное отображение. В качестве применения полученных результатов на группе Гейзенберга $\mathbb{H}^1$ вычислены коэффициенты Бельтрами некоторых квазиконформных отображений, и доказана теорема об образах квазиброуновских движений. В конкретных примерах мы демонстрируем инвариантность коэффициента Бельтрами под действием на соответствующее отображение слева композицией конформной функции. С помощью доказанной факторизации Стоилова на группе Гейзенберга, мы показали, что если у двух квазиброуновских движений их соответствующие коэффициенты Бельтрами равны почти всюду, то их траектории эквивалентны только в случае, если конформное отображение в факторизации Стоилова есть отображение, полученное из композиции сдвигов, поворотов и растяжений.

Ключевые слова: группа Гейзенберга, факторизация Стоилова, квазиконформные отображения, система Бельтрами, броуновское движение.

AMS Subject Classification: 53C17, 34C05.

Образец цитирования: *Dorokhin D. K.* Stoilow Factorization of the Heisenberg Group // Владикавк. мат. журн.—2025.—Т. 27, № 3.—С. 50–59 (in English). DOI: 10.46698/o8833-7719-4418-f.