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ON BAND PRESERVING OPERATORS ON COMPLEX VECTOR LATTICES[#]

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Dedicated to the memory of S. S. Kutateladze

Abstract. In this article we continue an investigation of orthogonally additive operators on complex vector lattices started in [1]. We study the special class of so called band preserving orthogonally additive operators defined on the complexification $E_{\mathbb{C}}$ of a uniformly complete vector lattice E and taking values in E. We say that an orthogonally additive operator $\mathscr{T}\colon E_{\mathbb{C}} \to E$ is band preserving if $\mathscr{T}(w) \in \{|w|\}^{\perp\perp}$ for every element w of $E_{\mathbb{C}}$. The authors introduce and study the class of elementary band preserving operators, which are complex extensions $\mathscr{T}_{T,S}$ constructed from pairs of real operators $T, S\colon E \to E$ that commute with all band projections. It is demonstrated that such operators are not only band preserving, but also regular. A central result of the work is that the set $\mathscr{N}(E_{\mathbb{C}}, E)$ of all elementary band preserving operators constitutes a vector sublattice within the Dedekind complete vector lattice $\mathscr{OA}_r(E_{\mathbb{C}}, E)$ of all regular orthogonally additive operators. The lattice operations in this sublattice are shown to be calculated pointwise, mirroring the structure of the target space E, with explicit formulas provided for the supremum, infimum, positive part, negative part, and modulus. Furthermore, it is established that $\mathscr{N}(E_{\mathbb{C}}, E)$ is contained within the band generated by the complex extension of the identity operator $\{\mathscr{T}_{I,I}\}^{\perp\perp}$.

Keywords: orthogonally additive operator, band preserving operator, regular operator, order projection, vector lattice, complexification.

AMS Subject Classification: 47H30, 47H99.

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1. Introduction

Linear disjointness preserving operators on vector and Banach lattices had been studied during the early part of the twentieth century (see the survey article [2] and references therein). Orthogonally additive (in general nonlinear) operators (OAOs) on vector lattices were introduced in [3]. Some deep results on different classes of orthogonally additive operators on vector and Banach lattices were obtained in [4–9]. Disjointness preserving OAOs on vector lattices were studied in [10–14].

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In [1] the concept of an orthogonally additive operator was extended to the setting of maps defined on the complexification $E_{\mathbb{C}}$ of a uniformly complete vector lattice E and taking values in a Dedekind complete vector lattice F. It was proved in [1] that the vector space $\mathcal{OA}_r(E_{\mathbb{C}}, F)$ of all regular orthogonally additive operators from $E_{\mathbb{C}}$ to a Dedekind complete vector lattice F is a Dedekind complete vector lattice with respect to the natural partial order on $\mathcal{OA}_r(E_{\mathbb{C}}, F)$.

In this paper we continue this line of thought. We study band preserving OAOs defined on $E_{\mathbb{C}}$ and taking values in E. The article is organized as follows. In the next section, we present the necessary information on complex vector lattices and orthogonally additive operators. Then we introduce band preserving orthogonally additive operators from $E_{\mathbb{C}}$ to E and present some basic examples of such operators. We show that with a pair of orthogonally additive operators $T, S\colon E \to E$ is associated the orthogonally additive operator $\mathcal{T}_{T,S}\colon E_{\mathbb{C}} \to E$ which called the complex extension of T and S. We prove that a complex extension $\mathcal{T}_{T,S}\colon E_{\mathbb{C}} \to E$ of commuting with projections operators $T, S\colon E \to E$ is a band preserving operator (Proposition 3.2). We consider a special class of a band preserving operators $\mathcal{T}\colon E_{\mathbb{C}} \to E$, which have the form $\mathcal{T} = \mathcal{T}_{T,S}$, where $T, S\colon E \to E$ are commuting with projections operators. We call these operators elementary band preserving operators. We prove that the set $\mathcal{N}(E_{\mathbb{C}}, E)$ of all elementary band preserving operators is a vector sublattice of $\mathcal{OA}_r(E_{\mathbb{C}}, E)$ and there is the inclusion $\mathcal{N}(E_{\mathbb{C}}, E) \subset \{I_{\mathbb{C}}\}^{\perp\perp}$ (Theorem 3.1). Finally, two open problems are stated.

2. Preliminaries

In this section we present some necessary facts and notations that we need in the sequel. For the standard information on the theory of vector lattices and regular linear operators between them we refer the reader to [15–17]. All vector lattices we consider below are supposed to be Archimedean. The identity operator on a vector space W we denote by I_W. The term «operator from vector spaces E and F» means an arbitrary map $\mathcal{T}\colon E \to F$.

Two elements e, f of a vector lattice E are called *disjoint* (notation $e \perp f$), if $|e| \wedge |f| = 0$. The sum $e + f$ of disjoint elements e and f we denote by $e \sqcup f$.

Given a net $(e_\alpha)_{\alpha\in A}$ in a vector lattice E *order converges* to $e \in E$, if there exists a net $(f_\xi)_{\xi\in\Xi}$ in E_+, such that $f_\xi \downarrow 0$ and for every $\xi \in \Xi$ there is an index $\alpha(\xi) \in A$, such that $|e - e_\alpha| \leqslant f_\xi$ for all $\alpha \geqslant \alpha(\xi)$. We note that for a Dedekind complete vector lattice E a net $(e_\alpha)_{\alpha\in A}$ in E order converges to e if and only if there exists a net $(f_\alpha)_{\alpha\in A}$ in E_+, such that $f_\alpha \downarrow 0$ and $|e - e_\alpha| \leqslant f_\alpha$ for all $\alpha \geqslant \alpha_0$, with some $\alpha_0 \in A$. A linear subspace I of a vector lattice E is called an *order ideal* of E, if for every $x \in I$ and $y \in E$ the relation $|y| \leqslant |x|$ implies that $y \in I$. We note that every order ideal I of E is a vector sublattice of E. An order ideal I of E is said to be *order closed* if for every net $(e_\alpha)_{\alpha\in A}$ in I, which order converges to an element $e \in E$, it follows that $e \in I$. An order closed order ideal I of a vector lattice E is called a *band*. Consider a subset A of a vector lattice E. By A^d is denoted the set

$$A^d := \bigl\{e \in E : e \perp x, \forall\, x \in A\bigr\}.$$

As usual $A^{dd} = (A^d)^d$. It is well known that A^d is a band of E [8, p. 34]. A band $\mathcal{B}$ of vector lattice E is said to be a *projection band*, if

$$E = \mathcal{B} \oplus \mathcal{B}^d.$$

Suppose that $\mathcal{B}$ is a projection band in E. Then every element $e \in E$ has a unique decomposition $e = e_1 \sqcup e_2$, where $e_1 \in \mathcal{B}$ and $e_2 \in \mathcal{B}^d$ and there exists a positive projection

$\pi_{\mathscr{B}}\colon E \to E$ defined by the formula $\pi_{\mathscr{B}}e = e_1$. A projection of the form $\pi_{\mathscr{B}}$ is called an *order projection* (or a band projection onto the band $\mathscr{B}$). We say that E is a vector lattice with the *principal projection property* if $\{e\}^{dd}$ is a projection band for all $e \in E$. The order projection in E onto the band $\{e\}^{dd}$ is denoted by π_e.

The set of all order projections on E is denoted by $\mathfrak{B}(E)$. There is a natural partial order on $\mathfrak{B}(E)$, namely $\pi \leqslant \rho \Leftrightarrow \pi \circ \rho = \pi$. We note that the partially ordered set $\mathfrak{B}(E)$ is actually a Boolean algebra with respect to the Boolean operations:

$$\pi \wedge \rho := \pi \circ \rho; \quad \pi \vee \rho := \pi + \rho - \pi \circ \rho; \quad \overline{\pi} = I - \pi.$$

We say that an element f of a vector lattice E is a *fragment* of $e \in E$, and use the notation $f \sqsubseteq e$, if $f \perp (e - f)$. The set of all fragments of an element $e \in E$ is denoted by $\mathfrak{F}_e$. The relation $\sqsubseteq$ is turned out to be a partial order on E which is called the *lateral order* (see [18]).

Definition 2.1. A sequence $(e_n)_{n\in\mathbb{N}}$ in a vector lattice E is said to be *uniformly Cauchy* whenever there exists some $e \in E_+$, such that for every $\varepsilon > 0$ the inequality $|e_n - e_m| \leqslant \varepsilon e$ holds for all sufficiently large n and m. We say that a vector lattice E is *uniformly complete* whenever every uniformly Cauchy sequence is relatively uniformly convergent.

We observe that every Dedekind σ-complete vector lattice is uniformly complete [8, p. 111].

Definition 2.2. Let X be a real vector space. We say that a complex vector space $X_{\mathbb{C}}$ defined by

$$X_{\mathbb{C}} := X + iX = \big\{x + iy : \ x, y \in X\big\}$$

is a *complexification* of X. The vector space operations on $X_{\mathbb{C}}$ defined by

$$(x + iy) + (v + iu) = x + v + i(y + u) \quad \text{and} \quad (\alpha + i\beta)(x + iy) = \alpha x - \beta y + i(\alpha y + \beta x)$$
$$\text{for all } \ \alpha, \beta \in \mathbb{R} \ \text{ and } \ x, y, v, u \in X.$$

Proposition 2.1 [19, Proposition 2.2.1]**.** *Suppose E is a uniformly complete vector lattice. Then for every $z = x + iy \in E_{\mathbb{C}}$ the following supremum*

$$|z| = |x + iy| := \sup_{0\leqslant\varphi\leqslant 2\pi} \big\{(\cos\varphi)x + (\sin\varphi)y\big\}$$

exists in E_+ which is called the modulus of z. Moreover the modulus possesses the following properties:

1) $|z| = 0 \Leftrightarrow z = 0$;
2) $|\lambda z| = |\lambda|\,|z|$ *for all* $\lambda \in \mathbb{C}$ *and* $z \in E_{\mathbb{C}}$;
3) $|z + w| \leqslant |z| + |w|$ *for all* $z, w \in E_{\mathbb{C}}$.

Remark 2.1. It is worth noting that

$$|x| \vee |y| \leqslant |z| \leqslant |x| + |y| \ \text{ for every } \ x + iy = z \in E_{\mathbb{C}}.$$

Definition 2.3 [20, Definition 3.1]. Suppose that E is a uniformly complete vector lattice. Two elements $z, w \in E_{\mathbb{C}}$ are said to be *disjoint* (notation $z \perp_{\mathbb{C}} w$) if $|z| \wedge |w| = 0$. An element $w \in E_{\mathbb{C}}$ is called a *fragment* of z if $(z - w) \perp_{\mathbb{C}} w$. The set of all fragments of z is denoted by $\mathfrak{F}_z$. We shall write $w \sqsubseteq_{\mathbb{C}} z$, if $w \in \mathfrak{F}_z$. We write $z = \bigsqcup_{i=1}^{n} z_i$, if $z = \sum_{i=1}^{n} z_i$ and $z_i \perp_{\mathbb{C}} z_j$ for all $i \neq j$. In particular, if $n = 2$, as in the case of vector lattices, we use the notation $z = z_1 \sqcup z_2$.

Proposition 2.2 [20, Proposition 3.1]. *Let E be a uniformly complete vector lattice, $w, v \in E_{\mathbb{C}}$, $w = x + iy$ and $v = f + ig$. Then the following statements are equivalent:*

1) $w \perp_{\mathbb{C}} v$;
2) $(|x| + |y|) \perp (|f| + |g|)$.

Proposition 2.3 [20, Proposition 3.2]. *Let E be a uniformly complete vector lattice. Then $\sqsubseteq_{\mathbb{C}}$ is a partial order on $E_{\mathbb{C}}$.* As in the case of real vector lattices the relation $\sqsubseteq_{\mathbb{C}}$ is called the *lateral order*. It was introduced and studied in [20].

Proposition 2.4 [20, Theorem 3.5]. *Let E be a uniformly complete vector lattice and $z = x + iy \in E_{\mathbb{C}}$. Then $\mathfrak{F}_z$, the set of all fragments of an element $z \in E_{\mathbb{C}}$, is a Boolean algebra with respect to the partial order $\sqsubseteq_{\mathbb{C}}$. Moreover $\mathfrak{F}_z$ is isomorphic to the Boolean subalgebra $\mathfrak{A}_{x,y}$ of $\mathfrak{F}_x \times \mathfrak{F}_y$ defined by*

$$\mathfrak{A}_{x,y} := \big\{(f, g) \in \mathfrak{F}_x \times \mathfrak{F}_y : \ f \perp (y - g) \text{ and } g \perp (x - f)\big\}.$$

Definition 2.4 [1, Definition 3.9]. Let E be a uniformly complete vector lattice and X be a vector space. A map $\mathscr{T} \colon E_{\mathbb{C}} \to X$ (not necessarily linear, not homogeneous) is said to be *orthogonally additive operator*, if

$$\mathscr{T}(u \sqcup v) = \mathscr{T}u + \mathscr{T}v \ \text{ for all disjoint } \ u, v \in E_{\mathbb{C}}.$$

Definition 2.5 [1, Definition 3.15]. Let E be a uniformly complete vector lattice and F be a vector lattice. An orthogonally additive operator $\mathscr{T} \colon E_{\mathbb{C}} \to F$ is called:

1) *positive*, if $\mathscr{T}z \geqslant 0$ holds in F for all $z \in E_{\mathbb{C}}$;
2) *regular*, if $\mathscr{T} = \mathscr{S}_1 - \mathscr{S}_2$, where $\mathscr{S}_1, \mathscr{S}_2$ are positive orthogonally additive operators from $E_{\mathbb{C}}$ to F;
3) *C-bounded*, if the set $T(\mathscr{F}_z)$ is order bounded in F for every $z \in E_{\mathbb{C}}$.

The sets of all positive, regular, and C-bounded orthogonally additive operators from $E_{\mathbb{C}}$ to F are denoted by $\mathscr{OA}_+(E_{\mathbb{C}}, F)$, $\mathscr{OA}_r(E_{\mathbb{C}}, F)$, and $\mathscr{OA}_{Cb}(E_{\mathbb{C}}, F)$ respectively.

Proposition 2.5 [1, Theorem 3.16]. *Let E be a uniformly complete vector lattice with the principal projection property and F be a Dedekind complete vector lattice. Then $\mathscr{OA}_{Cb}(E_{\mathbb{C}}, F) = \mathscr{OA}_r(E_{\mathbb{C}}, F)$ and $\mathscr{OA}_{Cb}(E_{\mathbb{C}}, F)$ is a Dedekind complete vector lattice. Moreover, the lattice operations on $\mathscr{OA}_{Cb}(E_{\mathbb{C}}, F)$ can be calculated by the following formulas:*

1) $(\mathscr{T} \vee \mathscr{S})z := \sup\{\mathscr{T}u + \mathscr{S}v : z = u \sqcup v\}$;
2) $(\mathscr{T} \wedge \mathscr{S})z := \inf\{\mathscr{T}u + \mathscr{S}v : z = u \sqcup v\}$;
3) $\mathscr{T}^+z := \sup\{\mathscr{T}u : u \sqsubseteq_{\mathbb{C}} z\}$;
4) $\mathscr{T}^-z := -\inf\{\mathscr{T}u : \ u \sqsubseteq_{\mathbb{C}} z\}$;
5) $|\mathscr{T}|z := \sup\{\mathscr{T}u - \mathscr{T}v : z = u \sqcup v\}$;
6) $|\mathscr{T}z| \leqslant |\mathscr{T}|z$

for every $\mathscr{S}, \mathscr{T} \in \mathscr{OA}_{Cb}(E_{\mathbb{C}}, F)$ and every $z \in E_{\mathbb{C}}$.

Let (A, Σ, μ) be a σ-finite measure space. The vector lattice of all equivalence classes of measurable real-valued functions on A is denoted by $L_0(\mu)$. Given $f \in L_0(\mu)_{\mathbb{C}}$ by $\operatorname{supp} f$, we denote the measurable set

$$\operatorname{supp} f := \big\{t \in A : \ f(t) \neq 0\big\}.$$

The characteristic function of a set D is denoted by 1_D. We recall that $H, D \in \Sigma$ are called *disjoint*, if $\mu\{t \in H \cap D\} = 0$. The union $H \cup D$ of two disjoint sets $H, D \in \Sigma$ we denote by $H \sqcup D$.

3. Band Preserving Operators

In this section we introduce band preserving operators on complex vector lattices and explore some of their properties. In particular, we consider real band preserving OAOs and prove the main result of these notes, concerning order properties of these operators (Theorem 3.1).

Definition 3.1. Let E be a uniformly complete vector lattice and $\mathscr{T}\colon E_{\mathbb{C}} \to E$ be an operator from $E_{\mathbb{C}}$ to E. We say that $\mathscr{T}$ is:

1) *a band preserving operator*, if $\mathscr{T}(w) \in \{|w|\}^{\perp\perp}$ for every $w \in E_{\mathbb{C}}$;

2) *disjointness preserving*, if $\mathscr{T}(w) \perp \mathscr{T}(v)$ for every $w, v \in E_{\mathbb{C}}$, such that $w \perp_{\mathbb{C}} v$.

It is clear that a band preserving operator $\mathscr{T}\colon E_{\mathbb{C}} \to E$ preserves disjointness.

Consider some examples of band preserving orthogonally additive operators.

Example 3.1 [1, Proposition 3.11]. Let E be a uniformly complete vector lattice. Then the modulus $|\cdot| : E_{\mathbb{C}} \to E$ is a band preserving orthogonally additive operator.

Definition 3.2. Let (A, Σ, μ) be a finite measure space. We say that a function $N : A \times \mathbb{C} \to \mathbb{R}$ is:

1) *superpositionally measurable* (or *super-measurable* for brevity), if $N(\cdot\,, f(\cdot)) \in L_0(\mu)$ for every measurable function $f\colon A \to \mathbb{C}$;

2) *normalized*, if $N(t, 0) = 0$ for almost all $t \in A$.

Proposition 3.1 [1, Proposition 3.14]**.** *Let $N\colon A \times \mathbb{C} \to \mathbb{R}$ be super-measurable, normalized function. Then there is the orthogonally additive operator $T_N\colon L_0(\mu)_{\mathbb{C}} \to L_0(\mu)$ defined by $T_N(f)(\cdot) = N(\cdot, f(\cdot))$, $f \in L_0(\mu)_{\mathbb{C}}$.*

Actually, it was shown in the proof of [1, Proposition 3.14] that $T_N(f) \in \{|f|\}^{dd}$ and therefore T_N is a band preserving operator. The operator T_N is known in a literature as a nonlinear superposition operator or Nemytskij operator [21].

Suppose that E is a uniformly complete vector lattice and X is a real vector space. To a pair of orthogonally additive operators $T, S\colon E \to X$ there is associated a map $\mathscr{T}_{T,S}\colon E_{\mathbb{C}} \to X$ defined by

$$\mathscr{T}_{T,S}(x + iy) = Tx + Sy, \quad x, y \in E. \tag{1}$$

Example 3.2 [1, Proposition 3.12]. $\mathscr{T}_{T,S}$ is an orthogonally additive operator from $E_{\mathbb{C}}$ to X.

We say that $\mathscr{T}_{T,S}\colon E_{\mathbb{C}} \to X$ is the *complex extension* of operators $T, S\colon E \to X$.

Definition 3.3 [11, Definition 2]. Let E be a vector lattice with the principal projection property. We say that an operator $T\colon E \to E$ *commutes with projections*, if $T\pi = \pi T$ for every $\pi \in \mathfrak{B}(E)$.

We note that a commuting with projections operator $T\colon E \to E$ is automatically orthogonally additive [11, Proposition 1].

Proposition 3.2. *Let E be a vector lattice with the principal projection property and $T, S\colon E \to E$ be commuting with projections operators. Then $\mathscr{T}_{T,S}\colon E_{\mathbb{C}} \to X$ is a band preserving operator.*

$\lhd$ Fix $w = x + iy$ with $x, y \in E$. Then by Proposition 2.2 we have that $\{|w|\}^{dd} = \{|x| + |y|\}^{dd}$. Now we may write

$$\begin{aligned}\mathscr{T}_{T,S}w = \mathscr{T}_{T,S}(x + iy) &= Tx + Sy = T\pi_{(|x|+|y|)}x + S\pi_{(|x|+|y|)}y \\ &= \pi_{(|x|+|y|)}Tx + \pi_{(|x|+|y|)}Sy = \pi_{(|x|+|y|)}(Tx + Sy) = \pi_{(|x|+|y|)}\mathscr{T}_{T,S}w\end{aligned}$$

and therefore $\mathscr{T}_{T,S}w \in \{|w|\}^{dd}$. $\rhd$

Proposition 3.3. *Let E be a Dedekind complete vector lattice and $T, S\colon E \to E$ be operators commuting with projections. Then $\mathscr{T}_{T,S} \in \mathscr{OA}_r(E_{\mathbb{C}}, E)$.*

$\lhd$ By Proposition 2.5 it is enough to prove that $\mathscr{T}_{T,S}$ is a C-bounded operator. Fixing $w = (x + iy) \in E_{\mathbb{C}}$, we will show that $\mathscr{T}_{T,S}(\mathfrak{F}_w)$ is an order bounded subset of E. Take an element $v \in \mathfrak{F}_w$. By Proposition 2.4 $v = c + id$, where $c \in \mathfrak{F}_x$ and $d \in \mathfrak{F}_y$, $c \perp (y - d)$ and $d \perp (x - c)$. Since $T, S\colon E \to E$ are disjointness preserving operators we have that

$$|Tx| = |T(c \sqcup (x - c))| = |Tc \sqcup T(x - c)| = |Tc| \sqcup |T(x - c)| \geqslant |Tc|$$
$$\text{and analogously } |Sy| \geqslant |Sd|.$$

Thus, it follows that $|Tc| \leqslant |Tx|$ and $|Sd| \leqslant |Sy|$ for every $c \in \mathfrak{F}_x$ and $d \in \mathfrak{F}_y$ and therefore

$$\mathscr{T}_{T,S}v = \mathscr{T}_{T,S}(c + id) = Tc + Sd \leqslant |Tc| + |Sd| \leqslant |Tx| + |Sy|.$$

Hence, $\mathscr{T}_{T,S}$ is a C-bounded and consequently a regular orthogonally additive operator. $\rhd$

An operator $\mathscr{T}\colon E_{\mathbb{C}} \to E$ which has the form $\mathscr{T}_{T,S}$ for some commuting with projections operators $T, S\colon E \to E$ is called an *elementary band preserving* operator. The set of all elementary band preserving operators from $E_{\mathbb{C}}$ to E is denoted by $\mathscr{N}(E_{\mathbb{C}}, E)$. It is not hard to verify that $\mathscr{N}(E_{\mathbb{C}}, E)$ is a real vector space. We recall that $I_E\colon E \to E$ is the identity operator on E. By $\mathscr{T}_{I,I}\colon E_{\mathbb{C}} \to E$ we denote its complex extension, that is

$$\mathscr{T}_{I,I}(x + iy) = I_E x + I_E y = x + y, \quad x, y \in E.$$

Clearly, $\mathscr{T}_{I,I}$ is an elementary band preserving operator from $E_{\mathbb{C}}$ to E. Now we ready to state the main result of these notes.

Theorem 3.1. *Let E be a Dedekind complete vector lattice. Then $\mathscr{N}(E_{\mathbb{C}}, E)$ is a vector sublattice of the Dedekind complete vector lattice $\mathscr{OA}_r(E_{\mathbb{C}}, E)$ of all regular OAOs from $E_{\mathbb{C}}$ to E and for every $\mathscr{T}_{T,S}, \mathscr{T}_{R,G} \in \mathscr{N}(E_{\mathbb{C}}, E)$, $w \in E_{\mathbb{C}}$ the following equalities hold:*

1) $(\mathscr{T}_{T,S} \vee \mathscr{T}_{R,G})w = \mathscr{T}_{T,S}w \vee \mathscr{T}_{R,G}w$;
2) $(\mathscr{T}_{T,S} \wedge \mathscr{T}_{R,G})w = \mathscr{T}_{T,S}w \wedge \mathscr{T}_{R,G}w$;
3) $(\mathscr{T}_{T,S})^+ w = (\mathscr{T}_{T,S}w)^+$;
4) $(\mathscr{T}_{T,S})^- w = (\mathscr{T}_{T,S}w)^-$;
5) $|\mathscr{T}_{T,S}|w = |\mathscr{T}_{T,S}w|$.

Moreover $\mathscr{N}(E_{\mathbb{C}}, E) \subset \{\mathscr{T}_{I,I}\}^{\perp\perp}$.

$\lhd$ Taking into account Proposition 3.3 we deduce that $\mathscr{N}(E_{\mathbb{C}}, E)$ is a linear subspace of a Dedekind complete vector lattice $\mathscr{OA}_r(E_{\mathbb{C}}, E)$. Let $\mathscr{T}_{T,S}, \mathscr{T}_{R,G} \in \mathscr{N}(E_{\mathbb{C}}, E)$ and $v = (x + iy) \in E_{\mathbb{C}}$. Then by Proposition 2.5 we have

$$(\mathscr{T}_{T,S} \vee \mathscr{T}_{R,G})v = \sup\bigl\{\mathscr{T}_{T,S}w + \mathscr{T}_{R,G}z :\ v = w \sqcup z\bigr\} \geqslant \mathscr{T}_{T,S}v \vee \mathscr{T}_{R,G}v.$$

Let us prove the converse inequality. Fix $w = a + ib$ and $z = c + id$, such that $v = w \sqcup z$. By Proposition 2.2 we have that $(|a|+|b|) \perp (|c|+|d|)$. It follows that $\pi_{(|a|+|b|)}x = a$, $\pi_{(|a|+|b|)}y = b$, $\pi_{(|c|+|d|)}x = c$, and $\pi_{(|c|+|d|)}y = d$. Now we may write

$$\begin{gathered}
\mathscr{T}_{T,S}w + \mathscr{T}_{R,G}z = \mathscr{T}_{T,S}(a + ib) + \mathscr{T}_{R,G}(c + id) = Ta + Sb + Rc + Gd \\
= T\pi_{(|a|+|b|)}x + S\pi_{(|a|+|b|)}y + R\pi_{(|c|+|d|)}x + G\pi_{(|c|+|d|)}y = \pi_{(|a|+|b|)}Tx + \pi_{(|a|+|b|)}Sy \\
+ \pi_{(|c|+|d|)}Rx + \pi_{(|c|+|d|)}Gy = \pi_{(|a|+|b|)}(Tx + Sy) + \pi_{(|c|+|d|)}(Rx + Gy) = \pi_{(|a|+|b|)}\mathscr{T}_{T,S}v \\
+ \pi_{(|c|+|d|)}\mathscr{T}_{R,G}v \leqslant \pi_{(|a|+|b|)}(\mathscr{T}_{T,S}v \vee \mathscr{T}_{R,G}v) + \pi_{(|c|+|d|)}(\mathscr{T}_{T,S}v \vee \mathscr{T}_{R,G}v) = \mathscr{T}_{T,S}v \vee \mathscr{T}_{R,G}v.
\end{gathered}$$

Passing to the supremum in left-hand side of the above inequality over all disjoint decompositions $v = w \sqcup z$ we have that

$$(\mathscr{T}_{T,S} \vee \mathscr{T}_{R,G})v \leqslant \mathscr{T}_{T,S}v \vee \mathscr{T}_{R,G}v$$

and consequently $(\mathscr{T}_{T,S} \vee \mathscr{T}_{R,G})v \leqslant \mathscr{T}_{T,S}v \vee \mathscr{T}_{R,G}v$ for every $v \in E_{\mathbb{C}}$. Now, we get all lattice operations for elements of $\mathscr{N}(E_{\mathbb{C}}, E)$ as follows:

$$(\mathscr{T}_{T,S} \wedge \mathscr{T}_{R,G})v = -\big((-\mathscr{T}_{T,S}) \vee (-\mathscr{T}_{R,G})\big)v = -\big((-\mathscr{T}_{T,S}v) \vee (-\mathscr{T}_{R,G}v)\big) = \mathscr{T}_{T,S}v \wedge \mathscr{T}_{R,G}v;$$

$$(\mathscr{T}^{+}_{T,S})v = (\mathscr{T}_{T,S} \vee 0)v = \mathscr{T}_{T,S}v \vee 0 = (\mathscr{T}_{T,S}v)^{+};$$

$$(\mathscr{T}_{T,S})^{-}v = -(\mathscr{T}_{T,S} \vee 0)v = -\mathscr{T}_{T,S}v \vee 0 = (\mathscr{T}_{T,S}v)^{-};$$

$$|\mathscr{T}_{T,S}|v = \big(\mathscr{T}_{T,S} \vee (-\mathscr{T}_{T,S})\big)v = \mathscr{T}_{T,S}v \vee (-\mathscr{T}_{T,S}v) = |\mathscr{T}_{T,S}v|.$$

Finally we show that $\mathscr{N}(E_{\mathbb{C}}, E) \subset \{\mathscr{T}_{I,I}\}^{\perp\perp}$. By [17, Theorem 4.3.4] it is enough to prove that $\mathscr{T}_{T,S} = \sup_n\{n|\mathscr{T}_{I,I}| \wedge \mathscr{T}_{T,S}\}$ for every positive $\mathscr{T}_{T,S} \in \mathscr{N}(E_{\mathbb{C}}, E)$. Take $0 \leqslant \mathscr{T}_{T,S} \in \mathscr{N}(E_{\mathbb{C}}, E)$ and $w \in E_{\mathbb{C}}$. By above we have that

$$\sup_n \big\{\big(n|\mathscr{T}_{I,I}| \wedge \mathscr{T}_{T,S}\big)w\big\} = \sup_n \big\{n|\mathscr{T}_{I,I}|w \wedge \mathscr{T}_{T,S}w\big\} = \sup_n \big\{n|w| \wedge \mathscr{T}_{T,S}w\big\}.$$

Since $\mathscr{T}_{T,S}v \in \{|v|\}^{dd}$ for every $v \in E_{\mathbb{C}}$ by [17, Theorem 4.3.4] it follows that $\mathscr{T}_{T,S}w = \sup_n\{n|\mathscr{T}_{I,I}|w \wedge \mathscr{T}_{T,S}w\}$ and the proof is finished. ▷

4. Open Problems

In this section we point two questions concerning band preserving operator from $E_{\mathbb{C}}$ to E.

PROBLEM 4.1: Does every band preserving operator $\mathscr{T} : E_{\mathbb{C}} \to E$ has the form $\mathscr{T} = \mathscr{T}_{T,S}$ for some commuting with projections operators $T, S : E \to E$?

We remark that in the real-case situation the vector space of all band preserving OAOs on E is the projection band of $\mathscr{OA}_r(E)$ which coincides with $\{I_E\}^{dd}$. Hence, the following natural problem arises.

PROBLEM 4.2: Is the inclusion $\mathscr{N}(E_{\mathbb{C}}, E) \subset \{\mathscr{T}_{I,I}\}^{\perp\perp}$ in Theorem 3.1 strict?

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О НЕРАСШИРЯЮЩИХ ОПЕРАТОРАХ В КОМПЛЕКСНЫХ ВЕКТОРНЫХ РЕШЕТКАХ

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Аннотация. Данная заметка продолжает цикл исследований, инициированных работой [1]. В статье рассматривается подкласс так называемых «нерасширяющих» ортогонально аддитивных операторов, заданных на комплексификации $E_{\mathbb{C}}$ равномерно полной векторной решетки и принимающих значения в E. Будем говорить, что ортогонально аддитивный оператор $\mathscr{T}\colon E_{\mathbb{C}} \to E$ является нерасширяющим, если $\mathscr{T}(w) \in \{|w|\}^{\perp\perp}$ для каждого элемента w из $E_{\mathbb{C}}$. Вводится и изучается класс элементарных нерасширяющих операторов, которые представляют собой комплексные расширения $\mathscr{T}_{T,S}$, построенные из пар вещественных операторов $T, S\colon E \to E$, коммутирующих со всеми нерасширяющими проекторами. Показано, что такие операторы не только являются нерасширяющими, но и регулярны. Представлено несколько примеров таких операторов и установлено, что действительное векторное пространство $\mathscr{N}(E_{\mathbb{C}}, E)$ всех элементарных нерасширяющих ортогонально аддитивных операторов является подрешеткой $\mathscr{OA}_r(E_{\mathbb{C}}, E)$ — порядково полной векторной решетки всех регулярных ортогонально аддитивных операторов из $E_{\mathbb{C}}$ в E. Показано, что операции решетки в этой подрешетке вычисляются поточечно, отражая структуру пространства E, с явными формулами для супремума, инфимума, положительной части, отрицательной части и модуля. Кроме того, установлено, что $\mathscr{N}(E_{\mathbb{C}}, E)$ содержится в полосе, порожденной комплексным расширением единичного оператора $\{\mathscr{T}_{I,I}\}^{\perp\perp}$.

Ключевые слова: ортогонально аддитивный оператор, нерасшираюший оператор, регулярный оператор, порядковый проектор, векторная решетка, комплексификация.

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