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EXISTENCE AND UNIQUENESS OF SOLUTION FOR NONLINEAR ANISOTROPIC ELLIPTIC DIRICHLET PROBLEMS

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Abstract. We consider boundary value problem for nonlinear anisotropic elliptic partial differential equations in bounded open Lipschitz domain and the Dirichlet boundary conditions. We also suppose that the body force function belongs to the natural dual space under certain hypotheses regarding the nonlinear anisotropic operators present on the main side of the proposed problems. We prove the existence and uniqueness of a weak solution in anisotropic Sobolev space for this problem. Our proofs are based on various anisotropic Sobolev inequalities, embedding theorems, and features of pseudo-monotone operators. The functional setting involves anisotropic Lebesgue and Sobolev spaces in the scalar case and their most important properties.

Keywords: nonlinear elliptic equations, anisotropic Sobolev spaces, weak solution, existence, uniqueness.

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1. Introduction

Our focus is on establishing both the existence and uniqueness of a weak solution for certain classes of nonlinear anisotropic elliptic equations of the following model:

$$\begin{cases} -\sum\limits_{i=1}^{N} \partial_i \left(|\partial_i u|^{p_i-2}\partial_i u\right) - \sum\limits_{i=1}^{N} \partial_i \left(\sigma_i(x,\partial_i u)\right) + \sum\limits_{i=1}^{N} u|u|^{p_i-2} = f(x), & \text{in } \Omega, \\ u = 0, & \text{in } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded open Lipschitz domain of $\mathbb{R}^N$, $N \geqslant 2$, $p_i > 1$, $i = 1, \ldots, N$, are real numbers, such that

$$\overline{p} < N \quad \text{and} \quad p_+ < \overline{p}^{\star}, \tag{2}$$

where $p_+ = \max_{i\in\{1,\ldots,N\}} p_i$, $\overline{p} = N\big(\sum_{i=1}^{N} \frac{1}{p_i}\big)^{-1}$ the harmonic mean of $\{p_1, \ldots, p_N\}$, with $\overline{p}^{\star} = \frac{N\overline{p}}{N-\overline{p}}$ denotes the Sobolev conjugate of $\overline{p}$, $f \in W_0^{-1,\overrightarrow{p}'}(\Omega)$ (i. e. belongs to the dual space of $W_0^{1,\overrightarrow{p}}(\Omega)$), with $\overrightarrow{p} = (p_1, \ldots, p_N)$, $\overrightarrow{p}' = (p_1', \ldots, p_N')$, and p_i' denotes the Hölder conjugate of p_i (i. e., $p_i' = \frac{p_i}{p_i-1}$, $i = 1, \ldots, N$), $\sigma_i : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$, $i = 1, \ldots, N$, are Carathéodory functions such that, for almost everywhere $x \in \Omega$, and every $\xi, \xi' \in \mathbb{R}$, $(\xi, \xi') \neq (0, 0)$:

$$\sigma_i(x,\xi)\xi \geqslant c_1|\xi|^{p_i}, \tag{3}$$

$$|\sigma_i(x,\xi)| \leqslant c_2\left(1+|\xi|^{p_i}\right)^{1-\frac{1}{p_i}}, \tag{4}$$

$$\left(\sigma_i(x,\xi)-\sigma_i(x,\xi')\right)\left(\xi-\xi'\right) \geqslant \begin{cases} c_3|\xi-\xi'|^{p_i}, & \text{if } p_i \geqslant 2,\\ c_4\frac{|\xi-\xi'|^2}{(|\xi|+|\xi'|)^{2-p_i}}, & \text{if } p_i \in (1,2), \end{cases} \tag{5}$$

where $c_j > 0$, $j = 1,\dots,4$.

The anisotropy of the problem is due to the fact that the growth of each partial derivative is controlled by a different power. The interest in studying such operators lies in their application in the mathematical modeling of physical and mechanical processes in anisotropic continua, among them modeling of image processing and electro-rheological fluids; we refer to [1–4] for more details.

Numerous studies have addressed the topic of proving the uniqueness of a weak solution after demonstrating its existence in anisotropic (or isotropic) Sobolev space (i. e., $W_0^{1,\overrightarrow{p}}(\Omega)$ or $W_0^{1,p}(\Omega)$) for non-linear elliptic problems with data belonging to the natural dual space and with different conditions. However, these studies were conducted under restrictions on the exponent p (or $\overrightarrow{p}$) of Sobolev space. In [5], the uniqueness of the weak solution is demonstrated when at least one $p_i \leqslant 2$, $i = 1,\dots,N$ and when the modulus of continuity of a_i, $i = 1,\dots,N$ (i. e., the coefficients of $|\partial_i u|^{p_i-2}\partial_i u$ for the main side of the problem) is globally controlled. For the isotropic scalar case, the uniqueness result in [6] to the equation $-\mathrm{div}(a(x,u,Du)) + \lambda u|u|^{p-2} = f$, fails for ($p > 2$, $\lambda = 0$), but it holds for ($1 < p \leqslant 2$, $\lambda \geqslant 0$) or ($2 < p < \infty$, $\lambda > 0$). The study in [7] for the equation $-\mathrm{div}(a(x,u)|\nabla u|^{p-2}\nabla u - \varphi(u)) = f$, also proved the existence only of the weak solution and the failure of the uniqueness when $p > 2$. To learn more about the works that studied the uniqueness of the solution; we can refer to [8–10].

In our paper, we have proved the existence and uniqueness of a weak solution to a nonlinear anisotropic elliptic problem with $W_0^{-1,\overrightarrow{p}'}(\Omega)$-data under the above hypotheses. We based our proof on the main theorem on pseudo-monotone operators (Theorem 27.A in [11] and see also [12–14] for the existence and on the various anisotropic Sobolev inequalities for the uniqueness.

The structure of our paper is as follows: Important definitions, characteristics, and ideas pertaining to $\overrightarrow{p}$-Sobolev spaces are covered in Section 2, along with a review of certain anisotropic Sobolev inequalities. Section 3 contains the main theorem and its proof.

2. Preliminaries

The most significant results, basic properties, and embedding theorems pertaining to anisotropic Sobolev spaces in the scalar case will be examined in this section; for that we can refer to [15–19].

Let $\Omega \subset \mathbb{R}^N$ ($N \geqslant 2$) be a bounded open Lipschitz domain. Let $1 < p_i < \infty$, $i = 1,\dots,N$ be real numbers. We set

$$\overrightarrow{p} = (p_1,\dots,p_N), \quad p_+ = \max_{i\in\{1,\dots,N\}} p_i, \quad p_- = \min_{i\in\{1,\dots,N\}} p_i,$$

$$\overline{p} = N\left(\sum_{i=1}^{N}\frac{1}{p_i}\right)^{-1} \quad \text{(the harmonic mean of } \{p_1,\dots,p_N\}),$$

$$\overline{p}^{\star} = \frac{N\overline{p}}{N-\overline{p}} \quad \text{(the Sobolev conjugate of } \overline{p}(<N)).$$

The anisotropic Sobolev space $W_0^{1,\overrightarrow{p}}(\Omega)$ is defined as follows:

$$W_0^{1,\overrightarrow{p}}(\Omega) = \left\{ u \in W_0^{1,1}(\Omega) : \ \partial_i u \in L^{p_i}(\Omega), \, i = 1, \ldots, N \right\}.$$

It is a reflexive Banach space when endowed with the following norm:

$$\|u\|_{\overrightarrow{p}} = \sum_{i=1}^{N} \|u\|_{L^{p_i}(\Omega)} + \sum_{i=1}^{N} \|\partial_i u\|_{L^{p_i}(\Omega)}. \tag{6}$$

The Banach space $\left(W_0^{1,\overrightarrow{p}}(\Omega), \|\cdot\|_{\overrightarrow{p}}\right)$ is defined as follows

$$W_0^{1,\overrightarrow{p}}(\Omega) = W^{1,\overrightarrow{p}}(\Omega) \cap W_0^{1,1}(\Omega) = \overline{C_0^{\infty}(\Omega)}^{W^{1,\overrightarrow{p}}(\Omega)}.$$

If $\overline{p} < N$, then the following embedding (see [18])

$$W_0^{1,\overrightarrow{p}}(\Omega) \hookrightarrow L^r(\Omega), \tag{7}$$

is continuous if $1 \leqslant r \leqslant \overline{p}^{\star}$ and also compact if $1 \leqslant r < \overline{p}^{\star}$. In addition, there exists $c > 0$ dependent on N, p_i, $i = 1 \ldots, N$ and $|\Omega|$, such that for all $u \in C_0^{\infty}(\Omega)$ and all $1 \leqslant r \leqslant \overline{p}^{\star}$

$$\|u\|_{L^r(\Omega)} \leqslant c \prod_{i=1}^{N} \|\partial_i u\|_{L^{p_i}(\Omega)}^{\frac{1}{N}}. \tag{8}$$

The relationship between the arithmetic and geometric means makes the previous Sobolev type inequality (8) implies that

$$\|u\|_{L^r(\Omega)} \leqslant C \sum_{i=1}^{N} \|\partial_i u\|_{L^{p_i}(\Omega)}. \tag{9}$$

If $\overline{p} \geqslant N$, then each of (7), (8), and (9) remains true for all $r \geqslant 1$. The following Poincaré type inequality (see [20, 21]) holds true

$$\|u\|_{L^{p_i}(\Omega)} \leqslant c \sum_{i=1}^{N} \|\partial_i u\|_{L^{p_i}(\Omega)}. \tag{10}$$

From (10) and (6) we conclude that

$$u \mapsto \sum_{i=1}^{N} \|\partial_i u\|_{L^{p_i}(\Omega)} \text{ is an equivalent norm to (6) on } W_0^{1,\overrightarrow{p}}(\Omega). \tag{11}$$

3. Existence and Uniqueness of Weak Solution

Definition 1. The function u is a weak solution to the equation (1) in $W_0^{1,\overrightarrow{p}}(\Omega)$ if and only if it satisfies for every $\phi \in W_0^{1,\overrightarrow{p}}(\Omega)$ the following:

$$\sum_{i=1}^{N} \int_{\Omega} \left(|\partial_i u|^{p_i - 2} \partial_i u + \sigma_i(x, \partial_i u) \right) \partial_i \phi \, dx + \sum_{i=1}^{N} \int_{\Omega} u|u|^{p_i-2} \phi \, dx = \langle f, \phi \rangle. \tag{12}$$

Our main result is the following theorem:

Theorem 1. *Let* $1 < p_1, \ldots, p_N < +\infty$ *be real numbers,* $f \in W_0^{-1,\overrightarrow{p}'}(\Omega)$*, and* σ_i*,* $i = 1, \ldots, N$*, be Carathéodory functions, such that* (2)–(5) *holds. Then there exists a unique weak solution to* (1) *in* $W_0^{1,\overrightarrow{p}}(\Omega)$*.*

$\lhd$ We consider the operator $\mathbf{T} : W_0^{1,\overrightarrow{p}}(\Omega) \longrightarrow W_0^{-1,\overrightarrow{p}'}(\Omega)$, such that

$$\mathbf{T} : u \mapsto \left(\phi \mapsto \sum_{i=1}^{N} \int_{\Omega} \big(|\partial_i u|^{p_i - 2} \partial_i u + \sigma_i(x, \partial_i u) \big) \partial_i \phi \, dx + \sum_{i=1}^{N} \int_{\Omega} u |u|^{p_i - 2} \phi \, dx \right).$$

Then we have, for all $u, \phi \in W_0^{1,\overrightarrow{p}}(\Omega)$,

$$\langle \mathbf{T}(u), \phi \rangle = \sum_{i=1}^{N} \int_{\Omega} \big(|\partial_i u|^{p_i - 2} \partial_i u + \sigma_i(x, \partial_i u) \big) \partial_i \phi \, dx + \sum_{i=1}^{N} \int_{\Omega} u |u|^{p_i - 2} \phi \, dx.$$

We will follow a method similar to that in [22–25] to prove the boundedness, coerciveness, and pseudo-monotonicity of $\mathbf{T}$.

Putting $\mathbf{T} = \mathbf{T_1} + \mathbf{T_2}$, such that

$$\langle \mathbf{T_1}(u), \phi \rangle = \sum_{i=1}^{N} \int_{\Omega} \big(|\partial_i u|^{p_i - 2} \partial_i u + \sigma_i(x, \partial_i u) \big) \, \partial_i \phi \, dx,$$

$$\langle \mathbf{T_2}(u), \phi \rangle = \sum_{i=1}^{N} \int_{\Omega} u |u|^{p_i - 2} \phi \, dx.$$

From (4), (9), and the use of Hölder's inequality, we obtain

$$\begin{aligned}
|\langle \mathbf{T_1}(u), \phi \rangle| &\leqslant \sum_{i=1}^{N} \int_{\Omega} \big| |\partial_i u|^{p_i - 2} \partial_i u \big| \, |\partial_i \phi| \, dx + \sum_{i=1}^{N} \int_{\Omega} |\sigma_i(x, \partial_i u)| |\partial_i \phi| \, dx \\
&\leqslant \sum_{i=1}^{N} \int_{\Omega} |\partial_i u|^{p_i - 1} |\partial_i \phi| \, dx + c_2 \sum_{i=1}^{N} \int_{\Omega} \left(1 + |\partial_i u|^{p_i}\right)^{1 - \frac{1}{p_i}} |\partial_i \phi| \, dx \\
&\leqslant \sum_{i=1}^{N} \big\| |\partial_i u|^{p_i - 1} \big\|_{p_i'} \|\partial_i \phi\|_{p_i} + c_2 \sum_{i=1}^{N} \Big\| \left(1 + |\partial_i u|^{p_i}\right)^{1 - \frac{1}{p_i}} \Big\|_{p_i'} \|\partial_i \phi\|_{p_i} \\
&\leqslant \sum_{i=1}^{N} \left[\left(\int_{\Omega} |\partial_i u|^{p_i} \, dx \right)^{1 - \frac{1}{p_i}} + c_2 \left(|\Omega| + \int_{\Omega} |\partial_i u|^{p_i} \, dx \right)^{1 - \frac{1}{p_i}} \right] \sum_{i=1}^{N} \|\partial_i \phi\|_{p_i} \\
&\leqslant C \left(1 + N|\Omega| + \sum_{i=1}^{N} \|\partial_i u\|_{L^{p_i}(\Omega)}^{p_i} \right)^{1 - \frac{1}{p_-}} \|\phi\|_{\overrightarrow{p}} \\
&\leqslant C \left(1 + N|\Omega| + \left(1 + \|u\|_{\overrightarrow{p}}^{p_+} \right) \right)^{1 - \frac{1}{p_-}} \|\phi\|_{\overrightarrow{p}} \leqslant C' \left(1 + \|u\|_{\overrightarrow{p}}^{p_+} \right)^{1 - \frac{1}{p_-}} \|\phi\|_{\overrightarrow{p}}.
\end{aligned} \tag{13}$$

From (10), and using Hölder's inequality, we obtain

$$
\begin{aligned}
|\langle \mathbf{T_2}(u),\phi\rangle| &\leqslant \sum_{i=1}^{N}\int_{\Omega}|u|^{p_i-1}|\phi|\,dx \leqslant \sum_{i=1}^{N}\|u\|_{p_i}^{p_i-1}\|\phi\|_{p_i} \\
&\leqslant \sum_{i=1}^{N}\|u\|_{p_i}^{p_i-1}\sum_{i=1}^{N}\|\phi\|_{p_i} \leqslant \left(1+\left(\sum_{i=1}^{N}\|u\|_{p_i}\right)^{p_+-1}\right)\sum_{i=1}^{N}\|\phi\|_{p_i} \\
&\leqslant \left(1+(Nc\|u\|_{\overrightarrow{p}})^{p_+-1}\right)(Nc'\|\phi\|_{\overrightarrow{p}}) \leqslant C\left(1+\|u\|_{\overrightarrow{p}}^{p_+-1}\right)\|\phi\|_{\overrightarrow{p}}.
\end{aligned} \tag{14}
$$

So, the boundedness of $\mathbf{T}$ is provided by (13) and (14).

After dropping the non-negative term $\langle \mathbf{T_2}(u),u\rangle$, and using (3), we deduce

$$
\begin{aligned}
\frac{\langle \mathbf{T}(u),u\rangle}{\|u\|_{\overrightarrow{p}}} &\geqslant \frac{\sum_{i=1}^{N}|\partial_i u|^{p_i}+\sum_{i=1}^{N}\int_{\Omega}\sigma_i(x,\partial_i u)\partial_i u\,dx}{\|u\|_{\overrightarrow{p}}} \geqslant \frac{c\sum_{i=1}^{N}\|\partial_i u\|_{L^{p_i}(\Omega)}^{p_i}}{\|u\|_{\overrightarrow{p}}} \\
&\geqslant \frac{c\sum_{i=1}^{N}\min\left\{\|\partial_i u\|_{L^{p_i}(\Omega)}^{p_-},\|\partial_i u\|_{L^{p_i}(\Omega)}^{p_+}\right\}}{\|u\|_{\overrightarrow{p}}} \geqslant \frac{c\sum_{i=1}^{N}\|\partial_i u\|_{L^{p_i}(\Omega)}^{p_-}-Nc}{\|u\|_{\overrightarrow{p}}} \\
&\geqslant \frac{c\left(\frac{1}{N}\sum_{i=1}^{N}\|\partial_i u\|_{L^{p_i}(\Omega)}\right)^{p_-}-Nc}{\|u\|_{\overrightarrow{p}}} = \frac{c}{N^{p_-}}\|u\|_{\overrightarrow{p}}^{p_--1}-\frac{Nc}{\|u\|_{\overrightarrow{p}}}.
\end{aligned} \tag{15}
$$

Hence, (15) implies that $\mathbf{T}$ is coercive.

Let $(u_n)_n \subset W_0^{1,\overrightarrow{p}}(\Omega)$ be a sequence, such that

$$u_n \rightharpoonup u \text{ in } W_0^{1,\overrightarrow{p}}(\Omega), \tag{16}$$

$$\limsup_{n\to\infty}\langle \mathbf{T}(u_n),u_n-u\rangle \leqslant 0. \tag{17}$$

After setting

$$
\begin{aligned}
\Phi_{(i,n)}^{(1)} &= \big(\sigma_i(x,\partial_i u_n)-\sigma_i(x,\partial_i u)\big)(\partial_i u_n-\partial_i u),\\
\Phi_{(i,n)}^{(2)} &= \big(|\partial_i u_n|^{p_i-2}\partial_i u_n-|\partial_i u|^{p_i-2}\partial_i u\big)(\partial_i u_n-\partial_i u),\\
\Phi_{(i,n)}^{(3)} &= \big(|u_n|^{p_i-2}u_n-|u|^{p_i-2}u\big)(u_n-u),
\end{aligned}
$$

we obtain that

$$
\begin{aligned}
\langle \mathbf{T}(u_n),u_n-u\rangle &= \sum_{i=1}^{N}\int_{\Omega}\Phi_{(i,n)}^{(1)}\,dx+\sum_{i=1}^{N}\int_{\Omega}\sigma_i(x,\partial_i u)(\partial_i u_n-\partial_i u)\,dx \\
&+\sum_{i=1}^{N}\int_{\Omega}\Phi_{(i,n)}^{(2)}\,dx+\sum_{i=1}^{N}\int_{\Omega}|\partial_i u|^{p_i-2}\partial_i u(\partial_i u_n-\partial_i u)\,dx \\
&+\sum_{i=1}^{N}\int_{\Omega}\Phi_{(i,n)}^{(3)}\,dx+\sum_{i=1}^{N}\int_{\Omega}|u|^{p_i-2}u(u_n-u)\,dx.
\end{aligned} \tag{18}
$$

Since $u_n \longrightarrow u$ in $L^r(\Omega)$, where $1 \leqslant r < \overline{p}^*$ (due (7) and (16)), and the boundedness of $(|u|^{p_i-2}u)$ in $L^{p_i'}(\Omega)$, we conclude

$$\sum_{i=1}^{N} \int_{\Omega} |u|^{p_i-2}u(u_n - u)\, dx \longrightarrow 0. \tag{19}$$

Since $\partial_i u_n \rightharpoonup \partial_i u$ weakly in $L^{p_i}(\Omega)$, then (7) gives us

$$\partial_i u_n \longrightarrow \partial_i u \quad \text{strongly in} \quad L^r(\Omega), \quad 1 \leqslant r < \overline{p}^*. \tag{20}$$

From (20), and the boundedness of both $(|\partial_i u|^{p_i-2}\partial_i u)$, $\sigma_i(x, \partial_i u)$ in $L^{p_i'}(\Omega)$, we get

$$\sum_{i=1}^{N} \int_{\Omega} \sigma_i(x, \partial_i u)(\partial_i u_n - \partial_i u)\, dx \longrightarrow 0, \tag{21}$$

$$\sum_{i=1}^{N} \int_{\Omega} |\partial_i u|^{p_i-2}\partial_i u(\partial_i u_n - \partial_i u)\, dx \longrightarrow 0. \tag{22}$$

By combining (17), (18), (19), (21), and (22), we deduce that

$$\limsup_{n\to+\infty} \left(\sum_{i=1}^{N} \int_{\Omega} \Phi^{(1)}_{(i,n)}\, dx + \sum_{i=1}^{N} \int_{\Omega} \Phi^{(2)}_{(i,n)}\, dx + \sum_{i=1}^{N} \int_{\Omega} \Phi^{(3)}_{(i,n)}\, dx \right) \leqslant 0. \tag{23}$$

Let us recall this inequality: for every $\alpha, \beta \in \mathbb{R}$ ($(\alpha, \beta) \neq (0,0)$) and every $p > 1$, we have

$$\left(|\alpha|^{p-2}\alpha - |\beta|^{p-2}\beta\right)(\alpha - \beta) \geqslant \begin{cases} 2^{2-p}|\alpha - \beta|^p, & \text{if } p \geqslant 2, \\ (p-1)\frac{|\alpha-\beta|^2}{(|\alpha|+|\beta|)^{2-p}}, & \text{if } p \in (1,2). \end{cases} \tag{24}$$

Through (23), (5), and (24), we get

$$\lim_{n\to+\infty} \sum_{i=1}^{N} \int_{\Omega} \Phi^{(1)}_{(i,n)}\, dx = 0, \tag{25}$$

$$\lim_{n\to+\infty} \sum_{i=1}^{N} \int_{\Omega} \Phi^{(2)}_{(i,n)}\, dx = 0, \tag{26}$$

$$\lim_{n\to+\infty} \sum_{i=1}^{N} \int_{\Omega} \Phi^{(3)}_{(i,n)}\, dx = 0. \tag{27}$$

Now, we set that

$$A_{(i,n)} = \partial_i u_n - \partial_i u, \quad B_{(i,n)} = |\partial_i u_n| + |\partial_i u|.$$

If $p_i \geqslant 2$, then by (5) we get

$$\int_{\Omega} |A_{(i,n)}|^{p_i}\, dx \leqslant \frac{1}{c_3} \int_{\Omega} \Phi^{(1)}_{(i,n)}\, dx. \tag{28}$$

If $1 < p_i < 2$, then by (5) and Hölder's inequality we obtain

$$\begin{aligned}
\int_\Omega |A_{(i,n)}|^{p_i}\,dx &\leqslant \int_\Omega \frac{|A_{(i,n)}|^{p_i}}{\left(B_{(i,n)}\right)^{\frac{p_i(2-p_i)}{2}}} \left(B_{(i,n)}\right)^{\frac{p_i(2-p_i)}{2}}\,dx \\
&\leqslant \left\| \frac{|A_{(i,n)}|^{p_i}}{\left(B_{(i,n)}\right)^{\frac{p_i(2-p_i)}{2}}} \right\|_{L^{\frac{2}{p_i}}(\Omega)} \times \left\| \left(B_{(i,n)}\right)^{\frac{p_i(2-p_i)}{2}} \right\|_{L^{\frac{2}{2-p_i}}(\Omega)} \\
&\leqslant \max\left\{ \left(\int_\Omega \frac{|A_{(i,n)}|^2}{\left(B_{(i,n)}\right)^{2-p_i}}\,dx \right)^{\frac{p_-}{2}}, \left(\int_\Omega \frac{|A_{(i,n)}|^2}{\left(B_{(i,n)}\right)^{2-p_i}}\,dx \right)^{\frac{p_+}{2}} \right\} \\
&\times \max\left\{ \left(\int_\Omega \left(B_{(i,n)}\right)^{p_i}\,dx \right)^{\frac{2-p_-}{2}}, \left(\int_\Omega \left(B_{(i,n)}\right)^{p_i}\,dx \right)^{\frac{2-p_+}{2}} \right\} \\
&\leqslant \frac{1}{c_4} \max\left\{ \left(\int_\Omega \Phi^{(1)}_{(i,n)}\,dx \right)^{\frac{p_-}{2}}, \left(\int_\Omega \Phi^{(1)}_{(i,n)}\,dx \right)^{\frac{p_+}{2}} \right\} \\
&\times \max\left\{ \left(\int_\Omega \left(B_{(i,n)}\right)^{p_i}\,dx \right)^{\frac{2-p_-}{2}}, \left(\int_\Omega \left(B_{(i,n)}\right)^{p_i}\,dx \right)^{\frac{2-p_+}{2}} \right\}.
\end{aligned} \tag{29}$$

Since u, $u_n \in W_0^{1,\overrightarrow{p}}(\Omega)$, then thanks to (25) we can pass to the limit when $n \longrightarrow +\infty$ in both (28), (29) and thus get

$$\lim_{n\to+\infty} \int_\Omega |A_{(i,n)}|^{p_i} = 0, \quad i \in \{1,\dots,N\}. \tag{30}$$

So, (30) implies, for every $i \in \{1,\dots,N\}$,

$$\partial_i u_n \longrightarrow \partial_i u \quad \text{strongly in} \quad L^{p_i}(\Omega) \quad \text{and a.e. in} \quad \Omega. \tag{31}$$

We can point out here that in a similar way using (26) and with the help of (24) we can also simply get (31). By (31) we have

$$\sigma_i(x,\partial_i u_n) \rightharpoonup \sigma_i(x,\partial_i u) \quad \text{weakly in} \quad L^{p_i'}(\Omega),$$

then we have, for every $\phi \in W_0^{1,\overrightarrow{p}}(\Omega)$

$$\sum_{i=1}^N \int_\Omega \sigma_i(x,\partial_i u_n)\partial_i\phi \longrightarrow \sum_{i=1}^N \int_\Omega \sigma_i(x,\partial_i u)\partial_i\phi. \tag{32}$$

On the other hand, as $\sigma_i(x,\partial_i u_n) \longrightarrow \sigma_i(x,\partial_i u)$ and $\partial_i u_n \longrightarrow \partial_i u$ a.e. in Ω, Fatou's Lemma implies that

$$\liminf_{n\to+\infty} \sum_{i=1}^N \int_\Omega \sigma_i(x,\partial_i u_n)\partial_i u_n \geqslant \sum_{i=1}^N \int_\Omega \sigma_i(x,\partial_i u)\partial_i u. \tag{33}$$

From (32) and (33), we deduce that

$$\liminf_{n\to+\infty}\sum_{i=1}^{N}\int \Omega\sigma_i(x,\partial_i u_n)(\partial_i u_n-\partial_i\phi)\geqslant\sum_{i=1}^{N}\int_\Omega \sigma_i(x,\partial_i u)(\partial_i u-\partial_i\phi). \tag{34}$$

With proof steps similar to proof (34), we can easily get

$$\liminf_{n\to+\infty}\sum_{i=1}^{N}\int_\Omega |\partial_i u_n|^{p_i-2}\partial_i u_n(\partial_i u_n-\partial_i\phi)\geqslant\sum_{i=1}^{N}\int_\Omega |\partial_i u|^{p_i-2}\partial_i u(\partial_i u-\partial_i\phi). \tag{35}$$

In a similar way to proof (31) by using (27) and with the help of (24) we can simply get

$$u_n\longrightarrow u \quad \text{strongly in} \quad L^{p_i}(\Omega) \quad \text{and a.e. in } \Omega. \tag{36}$$

From (36), we deduce that

$$|u_n|^{p_i-2}u_n\rightharpoonup |u|^{p_i-2}u \quad \text{weakly in} \quad L^{p_i'}(\Omega), \quad \text{and a.e. in } \Omega.$$

and this means that

$$\liminf_{n\to+\infty}\sum_{i=1}^{N}\int_\Omega |u_n|^{p_i-2}u_n(u_n-\phi)\geqslant\sum_{i=1}^{N}\int_\Omega |u|^{p_i-2}u(u-\phi). \tag{37}$$

We also have, for every $\phi\in W_0^{1,\overrightarrow{p}}(\Omega)$

$$\begin{aligned}\langle \mathbf{T}(u_n),u_n-\phi\rangle &= \sum_{i=1}^{N}\int_\Omega \sigma_i(x,\partial_i u_n)(\partial_i u_n-\partial_i\phi)\,dx\\ +\sum_{i=1}^{N}\int_\Omega |\partial_i u_n|^{p_i-2}\partial_i u_n(u_n-\phi)\,dx &+\sum_{i=1}^{N}\int_\Omega |u_n|^{p_i-2}u_n(u_n-\phi)\,dx.\end{aligned} \tag{38}$$

By combining (34), (35), (37) and (38), we get for every $\phi\in W_0^{1,\overrightarrow{p}}(\Omega)$

$$\liminf_{n\to\infty}\langle \mathbf{T}(u_n),u_n-\phi\rangle\geqslant\langle \mathbf{T}(u),u-\phi\rangle. \tag{39}$$

Finally, the pseudo-monotonicity of $\mathbf{T}$ is implied by (39). Thus, all conditions of the main theorem on pseudo-monotone operators have been satisfied. Therefore, we have proven that our problem (1) has at least one weak solution.

Now we prove the uniqueness of this solution. Let u_1, u_2 be two weak solutions of (1). We have

$$\sum_{i=1}^{N}\int_\Omega \left(|\partial_i u_1|^{p_i-2}\partial_i u_1+\sigma_i(x,\partial_i u_1)\right)\partial_i\phi\,dx+\sum_{i=1}^{N}\int_\Omega u_1|u_1|^{p_i-2}\phi\,dx=\langle f,\phi\rangle, \tag{40}$$

$$\sum_{i=1}^{N}\int_\Omega \left(|\partial_i u_2|^{p_i-2}\partial_i u_2+\sigma_i(x,\partial_i u_2)\right)\partial_i\phi\,dx+\sum_{i=1}^{N}\int_\Omega u_2|u_2|^{p_i-2}\phi\,dx=\langle f,\phi\rangle. \tag{41}$$

By taking $\varphi = u_1 - u_2$ as a test function in (40) and in (41), then subtracting the results side by side, we can deduce that

$$\sum_{i=1}^{N}\int\limits_{\Omega}\left(|\partial_i u_1|^{p_i-2}\partial_i u_1 - |\partial_i u_2|^{p_i-2}\partial_i u_2\right)(\partial_i u_1 - \partial_i u_2)\,dx$$
$$+\sum_{i=1}^{N}\int\limits_{\Omega}(\sigma_i(x,\partial_i u_1) - \sigma_i(x,\partial_i u_2))(\partial_i u_1 - \partial_i u_2)\,dx \tag{42}$$
$$+\sum_{i=1}^{N}\int\limits_{\Omega}\left(|u_1|^{p_i-2}u_1 - |u_2|^{p_i-2}u_2\right)(u_1 - u_2)\,dx = 0.$$

Since (24), we conclude for every $i \in \{1,\dots,N\}$ that

$$\left(|\partial_i u_1|^{p_i-2}\partial_i u_1 - |\partial_i u_2|^{p_i-2}\partial_i u_2\right)(\partial_i u_1 - \partial_i u_2) \geqslant 0, \tag{43}$$

$$\left(|u_1|^{p_i-2}u_1 - |u_2|^{p_i-2}u_2\right)(u_1 - u_2) \geqslant 0. \tag{44}$$

Also, since (5), we get for all $i \in \{1,\dots,N\}$ that

$$(\sigma_i(x,\partial_i u_1) - \sigma_i(x,\partial_i u_2))(\partial_i u_1 - \partial_i u_2) \geqslant 0. \tag{45}$$

From (43), (44), (45) and (42), we obtain for all $i \in \{1,\dots,N\}$ that

$$\int\limits_{\Omega}\left(|\partial_i u_1|^{p_i-2}\partial_i u_1 - |\partial_i u_2|^{p_i-2}\partial_i u_2\right)(\partial_i u_1 - \partial_i u_2)\,dx = 0, \tag{46}$$

$$\int\limits_{\Omega}(\sigma_i(x,\partial_i u_1) - \sigma_i(x,\partial_i u_2))(\partial_i u_1 - \partial_i u_2)\,dx = 0, \tag{47}$$

$$\int\limits_{\Omega}\left(|u_1|^{p_i-2}u_1 - |u_2|^{p_i-2}u_2\right)(u_1 - u_2)\,dx = 0. \tag{48}$$

Now, after setting, for every $i = 1,\dots,N$,

$$\Lambda_i = \int\limits_{\Omega}\left(|\partial_i u_1|^{p_i-2}\partial_i u_1 - |\partial_i u_2|^{p_i-2}\partial_i u_2\right)(\partial_i u_1 - \partial_i u_2)\,dx,$$

$$A_i = \partial_i u_1 - \partial_i u_2 \quad \text{and} \quad B_i = |\partial_i u_1| + |\partial_i u_2|,$$

and like the proof steps followed in both (28), (29), thanks to (24) we can obtain, for all $i = 1,\dots,N$

$$\int\limits_{\Omega}|\partial_i(u_1 - u_2)|^{p_i}\,dx \leqslant c\Lambda_i, \tag{49}$$

$$\int\limits_{\Omega}|\partial_i(u_1 - u_2)|^{p_i}\,dx$$
$$\leqslant \frac{1}{c_4}\max\left\{\left(\int\limits_{\Omega}(B_i)^{p_i}\,dx\right)^{\frac{2-p_-}{2}},\left(\int\limits_{\Omega}(B_i)^{p_i}\,dx\right)^{\frac{2-p_+}{2}}\right\}\times\max\left\{\Lambda_i^{\frac{p_i^-}{2}},\Lambda_i^{\frac{p_i^+}{2}}\right\}. \tag{50}$$

Since $u_1, u_2 \in W_0^{1,\overrightarrow{p}}(\Omega)$, (50) implies that

$$\int\limits_{\Omega} |\partial_i(u_1 - u_2)|^{p_i}\,dx \leqslant c' \max\left\{\Lambda_i^{\frac{p_i^-}{2}}, \Lambda_i^{\frac{p_i^+}{2}}\right\}. \tag{51}$$

By combining (49), (51), and (46), we get that

$$\int\limits_{\Omega} |\partial_i(u_1 - u_2)|^{p_i}\,dx = 0, \quad i = 1, \ldots, N. \tag{52}$$

Then, we conclude that

$$\left\|\partial_i(u_1 - u_2)\right\|_{p_i} = 0, \quad i = 1, \ldots, N. \tag{53}$$

Hence, we get that

$$\left\|u_1 - u_2\right\|_{\overrightarrow{p}} = 0, \quad i = 1, \ldots, N. \tag{54}$$

Then, (54) implies that $u_1 = u_2$. Thus, Theorem 1 has been proven. ▷

Remark 1. In a similar way, we can prove the uniqueness of the solution by adopting one of the two equations (47) (with the help of (5)) or (48) (with the help of (24)) instead of (46).

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СУЩЕСТВОВАНИЕ И ЕДИНСТВЕННОСТЬ РЕШЕНИЯ ДЛЯ НЕЛИНЕЙНЫХ АНИЗОТРОПНЫХ ЭЛЛИПТИЧЕСКИХ ЗАДАЧ ДИРИХЛЕ

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Аннотация. Рассматривается краевая задача для нелинейных анизотропных эллиптических дифференциальных уравнений в частных производных в ограниченной открытой липшицевой области и граничными условиями Дирихле. При этом предполагается, что функция внешних сил принадлежит естественному двойственному пространству при определенных гипотезах относительно нелинейных анизотропных операторов, присутствующих в основной части предлагаемых задач. Центральный результат представляет собой доказательство существования и единственности слабого решения в анизотропном пространстве Соболева для этой задачи. Оно основывается на применении различных анизотропных неравенств Соболева, теорем вложения и определенных особенностях псевдомонотонных операторов. Отметим, что функциональная постановка задачи включает анизотропные пространства Лебега и Соболева в скалярном случае и их наиболее важные свойства.

Ключевые слова: нелинейные эллиптические уравнения, анизотропные пространства Соболева, слабое решение, существование, единственность.

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